

BOOLEAN COMPLEXES FOR FERRERS GRAPHS

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ABSTRACT. Ferrers graphs are bipartite graphs that correspond naturally to Ferrers shapes. In this paper, we determine the homotopy type of the boolean complex of Ferrers graphs. Previous work of the last two authors shows that the boolean complex of a graph is a wedge of spheres of maximal dimension. Thus the homotopy type of this complex depends entirely on the number of spheres in the wedge sum, called the boolean number of the Ferrers graph. Applying the results to staircase shapes, we show that the boolean numbers of the associated Ferrers graphs are the Genocchi numbers of the second kind, and obtain a relation between the Legendre-Stirling numbers and the Genocchi numbers of the second kind. In another application, we compute the boolean number of a complete bipartite graph, corresponding to a rectangular Ferrers shape, which is expressed in terms of the Stirling numbers of the second kind.

1. INTRODUCTION

Ferrers shapes, or Young shapes or partitions, are classical combinatorial objects arising in a variety of contexts including Schubert varieties, symmetric functions, hypergeometric series, permutation statistics, quantum mechanical operators, and inverse rook problems (see references in [1]). To such an object, one can relate a bipartite graph known as a *Ferrers graph*, as introduced in [4].

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition, where we require that $\lambda_1 \geq \dots \geq \lambda_r \geq 0$. The associated bipartite Ferrers graph has vertices $R \sqcup C$ where $R = \{x_1, \dots, x_r\}$ and $C = \{y_1, \dots, y_{\lambda_1}\}$, and there is an edge $\{x_i, y_j\}$ if and only if $x_i \geq j$. In particular, the degree of vertex x_i is λ_i . Figure 1 gives an example of a Ferrers graph and the associated Ferrers shape. In this paper, a Ferrers shape will have a prescribed number of rows, although some of these may be empty.

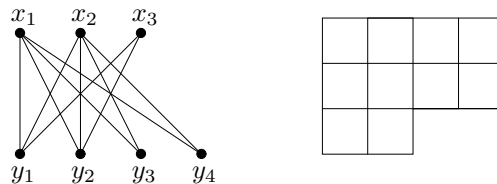


FIGURE 1. A Ferrers graph and its associated Ferrers shape. The corresponding partition is $\lambda = (4, 4, 2)$.

2000 *Mathematics Subject Classification*. Primary 05A15; Secondary 55P15, 05C99, 05A19.

Key words and phrases. Ferrers graph, Ferrers shape, partition, Legendre-Stirling numbers, Genocchi numbers of the second kind, Stirling number of the second kind, (complete) bipartite graph, homotopy type, boolean complex, boolean number.

A selection of enumerative properties of Ferrers graphs are studied in [4], where the graphs are introduced. In particular, expressions for the number of spanning trees, the number of Hamiltonian paths, the chromatic polynomial, and the chromatic symmetric function are given.

In [2], the authors find the number of independent sets of a Ferrers graph, and give relations between the set of independent sets of a Ferrers graph and certain combinatorial objects. Moreover, it is shown in [2] that the simplicial complex related to the set of independent sets of a non-rectangular Ferrers graph is simple-homotopic to a point, whereas it is simple-homotopic to two points in the case of a rectangular Ferrers graph.

Monomial and toric ideals associated to Ferrers graphs are studied in [1]. In particular, it is shown that the edge ideal of a Ferrers graph, called the Ferrers ideal, has a 2-linear minimal free resolution. That is, it defines a small subscheme, which is proved to characterize Ferrers graphs among bipartite graphs.

In this paper, we study the homotopy type of the boolean complexes of Ferrers graphs. Roughly speaking, the boolean complex of a graph G is the complex of words on the vertex set of G , without repeated letters, where two letters commute if the corresponding vertices are not connected by an edge in G . Boolean complexes were introduced in [7], where it is shown that the boolean complex of a graph G on n vertices always has the homotopy type of a wedge of spheres of dimension $n - 1$. The homotopy type is therefore determined by the number of spheres in the wedge sum, which we denote $m(G)$, and refer to as the *boolean number* of G .

The paper is organized as follows. In Section 2 we recall the definitions and results on Boolean complexes from [7] that will be needed in this paper, as well as some well-known sequences that arise in the course of the work. In Section 3 we provide a recursion for calculating the boolean number of an arbitrary Ferrers graph in terms of certain truncated shapes (Theorem 3.4). This formula is used in Section 4 to derive an identity (Theorem 4.3) for the boolean number in terms of certain coefficients $c_\lambda(n, j)$ that are defined recursively in the beginning of Section 4. More precisely, if F is the Ferrers graph associated to a Ferrers shape $\lambda = (\lambda_1, \dots, \lambda_r)$, then

$$m(F) = \sum_{j=0}^r c_\lambda(r, j) \cdot j^{\lambda_r}.$$

We restrict to Ferrers graphs for staircase shapes in Section 5, obtaining a connection to the Legendre-Stirling numbers. This, together with the above formula, enables the derivation of an explicit double sum formula for $m(F)$ in this case. A specialization to the (pure) staircase shape with $r \geq 1$ rows then gives that

$$m(F) = \sum_{j=1}^r \sum_{\ell=1}^j (-1)^{r+\ell} \frac{(2\ell+1)(\ell^2+\ell)^r j! j!}{(\ell+j+1)!(j-\ell)!}.$$

Intriguingly, these values, indexed by r , are the Genocchi numbers of the second kind. As a corollary to our considerations, we find a relationship between the Legendre-Stirling numbers $\{d(r, j)\}$ and the Genocchi numbers of the second kind $\{g(r)\}$ that seems to be new in the literature:

$$g(r) = \sum_{j=1}^r (-1)^{r+j} (j!)^2 \cdot d(r, j).$$

Applying Theorem 3.4 to complete bipartite graph $K_{r,k}$ in Section 6, which is really a specialization of Ferrers shapes to the rectangular shapes, yields

$$m(K_{r,k}) = \sum_{j=1}^r (-1)^{r-j} j! \left\{ \begin{matrix} r+1 \\ j+1 \end{matrix} \right\} j^k$$

where $\left\{ \begin{matrix} r+1 \\ j+1 \end{matrix} \right\}$ denotes a Stirling number of the second kind.

Finally, in Section 7, we derive a generating function related to the coefficients $c_\lambda(r, j)$ for Ferrers shapes with the differences between rows equal to a constant d , and sketch some directions for further research.

2. BACKGROUND MATERIAL

2.1. Boolean complexes. The main object of study in this paper is the boolean complex of a graph, as defined in [7]. Here we recall the basic definitions of boolean complexes pertaining to this paper, and the primary result about their homotopy type (Theorem 2.5). The reader is referred to [7] for a thorough discussion.

Definition 2.1. Let G be a finite simple graph with vertex set V . Construct a simplicial poset $\mathbb{B}(G)$ whose elements are equivalence classes of strings of distinct elements of V , where two strings are equivalent if one can be transformed into the other by commuting elements that are non-adjacent in G . The partial order on $\mathbb{B}(G)$ is induced by substring inclusion. To the poset $\mathbb{B}(G)$ we associate the regular cell complex $\Delta(G)$, called the *boolean complex* of G . The geometric realization of this complex is denoted $|\Delta(G)|$.

As discussed in [7], calculating the boolean number $m(G)$ of a graph uses three graph operations.

Definition 2.2. Let G be a finite simple graph, and let $s, t \in G$ be distinct vertices.

- If $\{s, t\}$ is an edge in G , then the graph $G - st$ is obtained from G by removing the edge $\{s, t\}$.
- G/st is the (simple) graph obtained from G by identifying s and t and deleting any resulting redundant edges and loops.
- $G \setminus st$ is the graph obtained from G by removing the vertices s and t , and all incident edges.

Definition 2.3. For a finite graph G , let $|G|$ denote the number of vertices in G . Also, for $n \geq 1$, let δ_n be the graph consisting of n disjoint points.

Definition 2.4. For integers $m, r \geq 0$, the notation $m \cdot S^r$ is used to indicate a wedge sum of m spheres of dimension r . In particular, $0 \cdot S^r$ is a single point.

We can now state the main theorem on the homotopy type of a boolean complex. The symbol \simeq is used in the statement to denote homotopy equivalence.

Theorem 2.5 ([7, Theorem 3.4]). *For every finite simple graph G , there is an integer $m(G)$ so that*

$$|\Delta(G)| \simeq m(G) \cdot S^{|G|-1}.$$

Moreover, the values $m(G)$ can be computed recursively by the following equations:

$$\begin{aligned} m(\delta_n) &= 0; \\ m(G) &= m(G - st) + m(G/st) + m(G \setminus st) \text{ if } \{s, t\} \text{ is an edge in } G. \end{aligned}$$

The convention $m(\emptyset) = 1$ is taken. The following result in [7] is often useful.

Proposition 2.6 ([7, Corollary 7.2]). *A finite simple graph G satisfies $m(G) = 0$ if and only if G has an isolated vertex.*

2.2. Sequences appearing in this paper. The *Legendre-Stirling numbers* are defined in [5] as

$$d(i, j) = \sum_{\ell=1}^j \frac{(-1)^{\ell+j} (2\ell+1)(\ell^2+\ell)^i}{(\ell+j+1)!(j-\ell)!}. \quad (1)$$

When $j > 1$ is fixed, the generating function of this sequence is

$$\sum_{i \geq 1} d(i, j) x^i = \frac{x^j}{\prod_{\ell=1}^j (1 - \ell(\ell+1)x)}.$$

These are sequence A071951 of [8].

The *Genocchi numbers of the second kind* (also known as the *median Genocchi numbers*) will be denoted $g(n)$ in this paper. They can be defined in several ways (see [8, A005439] and references therein). One interpretation is that $g(n)$ is the number of permutations $a_1 a_2 \cdots a_{2n+1}$ in S_{2n+1} such that $a_i > i$ if i is odd and $i < n$, and $a_i \leq i$ if i is even. These numbers also enumerate the boolean functions of n variables whose reduced ordered binary decision diagram contains exactly n branch nodes, one for each variable.

The well-known *Stirling numbers of the second kind* count the number of ways to partition an n element set into k nonempty blocks. It is sequence A008277 of [8].

3. RECURSION FOR A GENERAL FERRERS SHAPE

As mentioned previously, we make the convention that a Ferrers shape has a specified number of rows, even if some of these rows are empty. Such a Ferrers shape corresponds to a partition into a prescribed number of parts, where some parts are allowed to be zero.

Definition 3.1. For a Ferrers shape λ , the Ferrers graph associated to λ is denoted $F(\lambda)$. When no confusion will arise, the notation $m(\lambda)$ will be taken to mean $m(F(\lambda))$. If the shape λ has r rows and λ_1 columns, then the vertices of $F(\lambda)$ will be denoted $R \sqcup C$, where $R = \{x_1, \dots, x_r\}$ and $C = \{y_1, \dots, y_{\lambda_1}\}$, and there is an edge $\{x_i, y_j\}$ if and only if $\lambda_i \geq j$.

If λ has a row of length zero, that is, if some λ_i equals 0, then the corresponding vertex $x_i \in R$ has no incident edges. Consequently, the boolean number of such a graph is 0, by Proposition 2.6.

The aim of this section is to obtain a recursive formula for the boolean number of a Ferrers shape, based on the length of its bottom row. First we define the shapes appearing in the recursion.

Definition 3.2. For a Ferrers shape $\lambda = (\lambda_1, \dots, \lambda_r)$ with $r > 1$ rows, set $\underline{\lambda}$ to be the shape $(\lambda_1, \dots, \lambda_{r-1})$ having $r-1$ rows, obtained by deleting the bottom row from λ .

Definition 3.3. For a Ferrers shape $\lambda = (\lambda_1, \dots, \lambda_r)$ with r rows, and an integer $t \geq -\lambda_r$, set $\lambda[t]$ to be the shape

$$\lambda[t] = (\lambda_1 + t, \lambda_2 + t, \dots, \lambda_r + t),$$

having r rows.

The shape $\lambda[t]$ is obtained from λ by appending t columns of length r to the left side of the shape λ . If $t < 0$, then these columns are actually deleted from λ . When $t = -\lambda_r$, this means that all the boxes in the bottom row of λ are deleted, so the bottom row of $\lambda[-\lambda_r]$ is empty. Furthermore, for all i such that $\lambda_i = \lambda_r$, the i -th row of the shape $\lambda[-\lambda_r]$ is empty.

Theorem 3.4. *The boolean number of the Ferrers graph associated to the Ferrers shape $\lambda = (\lambda_1, \dots, \lambda_r)$, can be computed recursively according to the formula*

$$m(\lambda) = \begin{cases} 1 & \text{if } r = 1; \\ \lambda_r \cdot m(\underline{\lambda}) + \sum_{\ell=1}^{\lambda_r} \binom{\lambda_r+1}{\ell+1} \cdot m(\underline{\lambda}[-\ell]) & \text{if } r > 1. \end{cases} \quad (2)$$

Proof. If $r = 1$, then the graph is a star, and the boolean number of this graph was calculated in [7].

The remainder of the theorem is proved by induction on λ_r .

Let λ be as in the statement of the theorem and assume that $r > 1$. Set $F = F(\lambda)$ to be its Ferrers graph, and let the vertex $x_r \in F$ correspond to the last (r -th) row of λ . Thus the degree of x_r is λ_r . We apply the recursive procedure in Theorem 2.5 to the graph F and a specified edge in order to determine the boolean number $m(F) = m(\lambda)$. For our purposes, the specified edge is $\{x_r, y_{\lambda_r}\}$, which corresponds to the rightmost box of the bottom row in λ . We perform the three operations: the edge is deleted, the edge and its incident vertices are deleted, and the edge is contracted. The first two of these operations translate easily into the language of Ferrers graphs. More precisely, deleting the edge $\{x_r, y_{\lambda_r}\}$ corresponds to deleting the λ_r -th box from the r -th row of λ , which means subtracting 1 from the last part of the partition λ . Note that we still require that this shape have r rows, although the bottom row may be empty if $\lambda_r = 1$. Likewise, deleting the edge $\{x_r, y_{\lambda_r}\}$ and the incident vertices x_r and y_{λ_r} , along with their incident edges, corresponds to deleting the entire r -th row and λ_r -th column from λ , which gives the shape $\underline{\lambda}[-1]$.

Thus all that remains to understand is what happens when the edge $\{x_r, y_{\lambda_r}\}$ is contracted. Unfortunately, if $\lambda_r > 1$, the resulting graph is no longer bipartite, and so does not correspond to a Ferrers shape. However, if $\lambda_r = 1$, then contracting the edge $\{x_r, y_1\}$ yields the graph $\underline{\lambda}$. In this case, when $\lambda_r = 1$, the graph obtained after deleting the edge $\{x_r, y_1\}$ has an isolated vertex, which has boolean number 0. Thus, if $\lambda_r = 1$, then $m(\lambda)$ equals

$$m(\underline{\lambda}) + m(\underline{\lambda}[-1]),$$

which proves equation (2) in the base case.

Now suppose that the result has been proved when the last row of the shape has length less than λ_r . Then deleting the edge $\{x_r, y_{\lambda_r}\}$ contributes

$$m((\lambda_1, \dots, \lambda_{r-1}, \lambda_r - 1)) = (\lambda_r - 1) \cdot m(\underline{\lambda}) + \sum_{\ell=2}^{\lambda_r} \binom{\lambda_r}{\ell} \cdot m(\underline{\lambda}[-(\ell - 1)]),$$

to the boolean number, while deleting the edge $\{x_r, y_{\lambda_r}\}$ and its incident vertices contributes

$$m(\underline{\Delta}[-1]).$$

Combining these values gives the sum

$$(\lambda_r - 1) \cdot m(\underline{\Delta}) + \left(\binom{\lambda_r}{2} + 1 \right) \cdot m(\underline{\Delta}[-1]) + \sum_{\ell=3}^{\lambda_r} \binom{\lambda_r}{\ell} \cdot m(\underline{\Delta}[-(\ell-1)]). \quad (3)$$

For a Ferrers shape μ having a rows and b columns, and its associated Ferrers graph $F(\mu)$, let $F'(\mu)$ be the (likely non-bipartite) graph obtained from $F(\mu)$ by contracting the edge corresponding to the rightmost box in the bottom row of μ . With F as above and the graph $F'(\lambda)$ defined in this way, the boolean number $m(F)$ is equal to the sum of $m(F'(\lambda))$ and the expression in (3). We prove by induction on $\lambda_r \geq 2$ that $m(F'(\lambda))$ equals

$$m(\underline{\Delta}) + (\lambda_r - 1) \cdot m(\underline{\Delta}[-1]) + \sum_{\ell=2}^{\lambda_r} \binom{\lambda_r}{\ell} \cdot m(\underline{\Delta}[-\ell]). \quad (4)$$

The expression in (4) is straightforward to show if $\lambda_r = 2$, because the first term corresponds to deleting the edge between y_1 and y_2 in $F'(\lambda)$, the second term corresponds to contracting this edge, and the last term represents deleting the edge and its incident vertices. Now suppose inductively that the equality holds for all shapes whose last rows have fewer than λ_r boxes.

Deleting the edge $\{y_1, y_{\lambda_r}\}$ from the graph $F'(\lambda)$ yields the graph

$$F'((\lambda_1, \dots, \lambda_{r-1}, \lambda_r - 1)).$$

Likewise, deleting the edge $\{y_1, y_{\lambda_r}\}$ and its incident vertices y_1 and y_{λ_r} gives the graph $F(\underline{\Delta}[-2])$. Finally, contracting the edge $\{y_1, y_{\lambda_r}\}$ yields the graph $F'(\lambda[-1])$. Thus

$$m(F'(\lambda)) = m(F'((\lambda_1, \dots, \lambda_{r-1}, \lambda_r - 1))) + m(\underline{\Delta}[-2]) + m(F'(\lambda[-1])).$$

Thus $m(F'(\lambda))$ equals the expression in (4) by the inductive hypothesis and a binomial identity.

Finally, we combine the expressions in (3) and (4) to complete the proof. \square

4. FORMULA FOR GENERAL FERRERS SHAPE

In this section we obtain an explicit formula for the boolean number of a Ferrers graph.

Definition 4.1. For a Ferrers shape $\lambda = (\lambda_1, \dots, \lambda_r)$ with r rows, define the numbers $c_\lambda(i, j)$, where $1 \leq i \leq r$ and $j \in \mathbb{Z}$, recursively by

$$c_\lambda(1, j) = \begin{cases} -1 & \text{if } j = 0; \\ 1 & \text{if } j = 1; \\ 0 & \text{if } j \notin \{0, 1\}; \end{cases}$$

and

$$c_\lambda(i, j) = j(j-1)^{(\lambda_{i-1}-\lambda_i)} \cdot c_\lambda(i-1, j-1) - (j+1)^{(\lambda_{i-1}-\lambda_i)} \cdot c_\lambda(i-1, j),$$

for $1 < i \leq r$.

Here we use the convention $0^0 = 1$. Note that it follows directly from the definition that $c_\lambda(i, j)$ only takes nonzero values for $0 \leq j \leq i$, and so the values $c_\lambda(i, j)$ can be calculated by means of a triangle. The zero values $c_\lambda(i, j)$ for $j < 0$ or $j > i$ play no role in the paper, but are included in the definition so that we avoid exceptions in the recursive definition.

As an example, the triangle used to calculate $c_\lambda(i, j)$ for the Ferrers shape $\lambda = (7, 7, 7, 6, 4, 4, 2)$ is given in Table 1. This triangle exhibits three interesting phenomena, each of which can be shown to hold for any Ferrers shape. First the entries in the leftmost column are zero starting with the fourth row. In general one has $c_\lambda(i, 0) = 0$ when $\lambda_i < \lambda_1$. Second, when one disregards the leftmost columns, adjacent entries in the triangle have alternating signs. Third, the entries in each row sum to zero.

i	$j = 0$	1	2	3	4	5	6	7
1	-1	1						
2	1	-3	2					
3	-1	7	-12	6				
4	0	-14	86	-144	72			
5	0	28	-1060	6216	-10944	5760		
6	0	-56	3236	-28044	79584	-89280	34560	
7	0	112	-38944	1048416	-7376304	19758720	-22101120	8709120

TABLE 1. The triangle calculating $c_\lambda(i, j)$ for $\lambda = (7, 7, 7, 6, 4, 4, 2)$.

Lemma 4.2. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a Ferrers shape with r rows.*

- (a) $c_{\underline{\lambda}}(i, j) = c_\lambda(i, j)$, for all integers i and j with $1 \leq i \leq r - 1$.
- (b) If $t \geq -\lambda_r$, then $c_{\lambda[t]}(i, j) = c_\lambda(i, j)$ for all integers i and j with $1 \leq i \leq r$.

Proof. From the recursive definition of the numbers $c_\lambda(i, j)$, we see that they depend only on the differences $\lambda_{\ell-1} - \lambda_\ell$, and not on the individual values λ_ℓ . Removing the bottom row of λ to form $\underline{\lambda}$ does not change these differences for $\ell \leq r - 1$, proving part (a). Similarly, forming $\lambda[t]$ from λ does not change any differences, proving part (b). \square

Theorem 4.3. *The boolean number of the Ferrers graph associated to the Ferrers shape $\lambda = (\lambda_1, \dots, \lambda_r)$ is*

$$m(\lambda) = \sum_{j=0}^r c_\lambda(r, j) \cdot j^{\lambda_r}.$$

Proof. We prove this by induction on r using the recursive formula in Theorem 3.4. The base case, where $r = 1$, is easily checked.

To save notation we write $c(i, j)$ instead of $c_\lambda(i, j)$. This should not cause any confusion since, for $1 \leq i \leq r - 1$, and $\ell \leq \lambda_r$, we have

$$c_\lambda(i, j) = c_{\underline{\lambda}}(i, j) = c_{\underline{\lambda}[-\ell]}(i, j)$$

by Lemma 4.2.

Assuming that the result is true for shapes with at most $r - 1$ rows, we have the following string of equalities, where the first follows from equation (2):

$$\begin{aligned}
m(\lambda) &= \lambda_r \cdot m(\underline{\lambda}) + \sum_{\ell=1}^{\lambda_r} \binom{\lambda_r + 1}{\ell + 1} \cdot m(\underline{\lambda}[-\ell]) \\
&= \lambda_r \sum_{j=0}^{r-1} c(r-1, j) \cdot j^{\lambda_{r-1}} + \sum_{\ell=1}^{\lambda_r} \left(\binom{\lambda_r + 1}{\ell + 1} \cdot \sum_{j=0}^{r-1} c(r-1, j) \cdot j^{\lambda_{r-1}-\ell} \right) \\
&= \sum_{j=0}^{r-1} c(r-1, j) \left(\left(\sum_{\ell=0}^{\lambda_r} \binom{\lambda_r + 1}{\ell + 1} j^{\lambda_{r-1}-\ell} \right) - j^{\lambda_{r-1}} \right) \\
&= \sum_{j=0}^{r-1} c(r-1, j) \cdot \left((j^{\lambda_{r-1}-\lambda_r} ((j+1)^{\lambda_r+1} - j^{\lambda_r+1})) - j^{\lambda_{r-1}} \right) \\
&= \sum_{j=0}^{r-1} c(r-1, j) \cdot \left(j^{\lambda_{r-1}-\lambda_r} (j+1) \left((j+1)^{\lambda_r} - j^{\lambda_r} \right) \right). \tag{5}
\end{aligned}$$

If we recall that $c(r-1, -1) = c(r-1, r) = 0$, then rewriting the sum and collecting terms yields

$$\sum_{j=0}^r ((j-1)^{\lambda_{r-1}-\lambda_r} j \cdot c(r-1, j-1) - j^{\lambda_{r-1}-\lambda_r} (j+1) \cdot c(r-1, j)) \cdot j^{\lambda_r}$$

on the right hand side of (5), and as

$$(j-1)^{\lambda_{r-1}-\lambda_r} j \cdot c(r-1, j-1) - j^{\lambda_{r-1}-\lambda_r} (j+1) \cdot c(r-1, j) = c(r, j),$$

this completes the proof. \square

5. STAIRCASE SHAPES

In this section we specialize to staircase shapes:

Definition 5.1. For $r \geq 1$, a *staircase shape of height r* is the Ferrers shape $\sigma_r = (r, r-1, \dots, 1)$.

For a staircase shape σ_r , the recursive formula for $c_{\sigma_r}(i, j)$ simplifies to

$$c_{\sigma_r}(i, j) = j(j-1) \cdot c_{\sigma_r}(i-1, j-1) - (j+1)j \cdot c_{\sigma_r}(i-1, j).$$

Note, in particular, that $c_{\sigma_r}(i, 0) = 0$ for $i > 1$.

i	$j=0$	1	2	3	4	5	6	7
1	-1	1						
2	0	-2	2					
3	0	4	-16	12				
4	0	-8	104	-240	144			
5	0	16	-640	3504	-5760	2880		
6	0	-32	3872	-45888	157248	-201600	86400	
7	0	64	-23296	573888	-3695616	9192960	-9676800	3628800

TABLE 2. The triangle calculating $c_{\sigma(7)}(i, j)$.

Corollary 5.2. For $r \geq 1$, the values

$$\left\{ \frac{(-1)^{r+j}}{j!(j-1)!} \cdot c_{\sigma_r}(r, j) \right\} \quad (6)$$

are the Legendre-Stirling numbers.

Proof. The sequence in (6) has the same initial values and the same recurrence as the Legendre-Stirling numbers, given in [8, A071951]. \square

The Legendre-Stirling numbers are discussed in [5] and [6].

From the formula for the Legendre-Stirling numbers (see (1) in Section 2), one obtains a formula for the numbers $c_{\sigma_r}(r, j)$, and consequently for $m(\sigma_r)$ as well.

Corollary 5.3. For $r \geq 1$,

$$m(\sigma_r) = \sum_{j=1}^r \sum_{\ell=1}^j (-1)^{r+\ell} \frac{(2\ell+1)(\ell^2+\ell)^r \cdot j!j!}{(\ell+j+1)!(j-\ell)!}.$$

In fact, the boolean numbers of staircase shapes are equal to a known sequence.

Corollary 5.4. The sequence $\{m(\sigma_r)\}_{r \geq 1}$ is equal to the sequence of Genocchi numbers of the second kind.

Proof. The Genocchi numbers of the second kind can be calculated by $g(r) = G(r, 1)$, where $G(r, x)$ is the function defined recursively by

$$\begin{aligned} G(1, x) &= 1 \quad \text{and} \\ G(r, x) &= (x+1)^2 G(r-1, x+1) - x(x+1)G(r-1, x) \end{aligned}$$

for all $x \geq 0$ and $r \geq 2$.

By induction on i , one proves that

$$G(r, 1) = \sum_{j=1}^i j \cdot c_{\sigma_r}(i, j) \cdot G(r+1-i, j)$$

for $1 \leq i \leq r$. The base case $i = 1$ is trivial, and the inductive step follows easily from the recursive formulas for $G(r, x)$ and $c_{\sigma_r}(i, j)$. When $i = r$, we have $G(r+1-i, j) = 1$, and the equation simplifies to

$$g(r) = G(r, 1) = \sum_{j=1}^i j \cdot c_{\sigma_r}(r, j) = m(\sigma_r),$$

proving the claim. \square

The Genocchi numbers of the second kind were studied as early as the nineteenth century by Seidel. They are also treated more recently in [3].

Corollaries 5.2 and 5.4 reveal a relationship between the Legendre-Stirling numbers $\{d(r, j)\}$ and the Genocchi numbers of the second kind $\{g(r)\}$ that seems to be new in the literature.

Corollary 5.5. The Genocchi numbers of the second kind $\{g(r)\}$ and the Legendre-Stirling numbers $\{d(r, j)\}$ are related by the equation

$$g(r) = \sum_{j=1}^r (-1)^{r+j} (j!)^2 \cdot d(r, j).$$

6. COMPLETE BIPARTITE GRAPHS

Let k be a positive integer. In this section we consider the shape $\lambda = (\lambda_1, \dots, \lambda_r)$ with $\lambda_i = k$ for all $i = 1, \dots, r$. In other words, the shape λ is a rectangle having r rows and k columns. The corresponding Ferrers graph is the complete bipartite graph $K_{r,k}$. For a rectangular shape, the recursive formula for $c_\lambda(i, j)$ simplifies to

$$c_\lambda(i, j) = j \cdot c_\lambda(i-1, j-1) - (j+1) \cdot c_\lambda(i-1, j). \quad (7)$$

Proposition 6.1. *For positive integers r and k we have*

$$m(K_{r,k}) = \sum_{j=1}^r (-1)^{r-j} j! \left\{ \begin{matrix} r+1 \\ j+1 \end{matrix} \right\} j^k,$$

where $\left\{ \begin{matrix} r+1 \\ j+1 \end{matrix} \right\}$ denotes a Stirling number of the second kind.

Proof. Let $a(i, j) = (-1)^{i-j} j! \left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\}$. Trivially, $a(1, j) = c_\lambda(1, j)$. From the familiar recursion $\left\{ \begin{matrix} i+1 \\ j+1 \end{matrix} \right\} = \left\{ \begin{matrix} i \\ j \end{matrix} \right\} + (j+1) \cdot \left\{ \begin{matrix} i \\ j+1 \end{matrix} \right\}$ for the Stirling numbers it follows that

$$a(i, j) = j \cdot a(i-1, j-1) - (j+1) \cdot a(i-1, j).$$

Thus $a(i, j)$ and $c_\lambda(i, j)$ satisfy the same recursion. \square

7. DIRECTIONS FOR FURTHER RESEARCH

Having given a formula for the boolean number of an arbitrary Ferrers graph, several combinatorial questions emerge, as well as questions concerning the homotopy types of the boolean complexes for related graphs. First, it may be possible to simplify the formula of Theorem 4.3 for certain families of Ferrers shapes by calculating the coefficients $c_\lambda(r, j)$ using properties of the shape. Along these lines, given the appearances of the Legendre-Stirling numbers, the Genocchi numbers of the second kind, and the Stirling numbers of the second kind, we wonder if the boolean numbers of other families of Ferrers shape enumerate known combinatorial objects. Given a particular family of Ferrers shapes, we could also study the generating function of the boolean numbers of that family, indexed appropriately. Finally, it would be very interesting to extend this analysis of the homotopy type of the boolean complex of Ferrers graphs to some generalizations of these graphs. We address some of these questions in the remaining subsections.

7.1. Generating functions. The generating function for the Legendre-Stirling numbers $\{d(i, j)\}$, when $j > 1$ is fixed, is

$$\frac{x^j}{\prod_{\ell=1}^j (1 - \ell(\ell+1)x)}.$$

Recall that the Legendre-Stirling numbers are related to the numbers $\{c_{\sigma(r)}(i, j)\}$ that arise from the analysis of staircase shapes via the formula

$$d(i, j) = \frac{(-1)^{i+j}}{j!(j-1)!} \cdot c_{\sigma(r)}(i, j),$$

when $i \leq r$. In staircase shapes, the difference between consecutive parts is always equal to 1. As one would hope, we can generalize this statement to Ferrers shapes where consecutive parts always differ by a constant d .

Proposition 7.1. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a Ferrers shape such that $\lambda_{i-1} - \lambda_i = d$ for all $1 \leq i < r$. Put*

$$\widehat{c}_\lambda(i, j) = \frac{(-1)^{i+j} c_\lambda(i, j)}{j! ((j-1)!)^d}.$$

For $j \geq 1$, the sequence $\{\widehat{c}_\lambda(i, j)\}_{i \geq 1}$ has generating function

$$F_j(x) = \frac{x^j}{\prod_{i=1}^j (1 - i^d (i+1)x)}.$$

Proof. With the shape λ as in the statement of the proposition, the recursive formula for $c_\lambda(i, j)$ is

$$c_\lambda(i, j) = j(j-1)^d \cdot c_\lambda(i-1, j-1) - (j+1)j^d \cdot c_\lambda(i-1, j). \quad (8)$$

Multiplying equation (8) by $\frac{(-1)^{i+j}}{j!((j-1)!)^d}$ yields

$$\widehat{c}_\lambda(i, j) = \widehat{c}_\lambda(i-1, j-1) + (j+1)j^d \widehat{c}_\lambda(i-1, j). \quad (9)$$

Let the generating function for the sequence $\{\widehat{c}_\lambda(i, j)\}_{i \geq 1}$ be

$$F_j(x) = \sum_{i \geq 1} \widehat{c}_\lambda(i, j) x^i.$$

Multiplying both sides of equation (9) by x^i and summing over $i \geq 1$ gives

$$F_j(x) = xF_{j-1}(x) + (j+1)j^d xF_j(x),$$

which shows that

$$F_j(x) = \frac{x^j}{\prod_{i=1}^j (1 - i^d (i+1)x)}.$$

□

We state as an open question the following: What can be said concerning the generating function for $c_\lambda(r, j)$ where λ is an arbitrary shape? If the differences between parts varies, then one can no longer divide by a fixed product of factorials (such as the divisor $j!((j-1)!)^d$ above). This makes it harder to determine by which numbers one should divide the coefficients, if there even is a good choice. However, it may perhaps be possible to get a generating function for the coefficients $c_\lambda(i, j)$ without dividing by any factorials.

7.2. Generalizations of Ferrers graphs. One way, of many, to generalize the notion of a Ferrers graph is to consider a t -partite graph with vertices $R_1 \sqcup \dots \sqcup R_t$, where the subgraph induced by the vertices and edges of $R_i \sqcup R_{i+1}$ is a Ferrers graph for all $i = 1, \dots, t-1$. Call such a graph a t -Ferrers graph, where the 2-Ferrers graphs are the usual Ferrers graphs. The methods used in this paper do not have obvious analogs for t -Ferrers graphs with $t > 2$, and it would be of some interest to calculate these boolean numbers and thus understand the homotopy type of the relevant boolean complexes.

Another way to expand the results in this article would be to generalize Section 6 by determining the boolean number of the complete t -partite graph.

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