OBTAINABLE SIZES OF TOPOLOGIES ON FINITE SETS

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Abstract. We study the smallest possible number of points in a topological space having k open sets. Equivalently, this is the smallest possible number of elements in a poset having k order ideals. Using efficient algorithms for constructing a topology with a prescribed size, we show that this number has a logarithmic upper bound. We deduce that there exists a topology on n points having k open sets, for all k in an interval which is exponentially large in n. The construction algorithms can be modified to produce topologies where the smallest neighborhood of each point has a minimal size, and we give a range of obtainable sizes for such topologies.

1. Introduction

Finite topological spaces present many interesting combinatorial questions. The most fundamental of these concerns the number $T(n)$ of different topologies on n points. This number has been determined by exhaustive enumeration for $n \leq 16$ ([2]). The general question is very difficult, and it is uncertain whether a formula for $T(n)$ will ever be obtained, although Kleitman and Rothschild give asymptotic bounds in [3] and [4].

The enumeration of different topologies on n points can be refined by counting $T(n,k)$, the number of different topologies on n points having k open sets. Just as for $T(n)$, this is a long-standing open problem, although some special cases are known. In particular, Sharp [8] and Stephen [13] show that $T(n,k) = 0$ for all $k \in (3 \cdot 2^{n-2}, 2^n)$. The values of $T(n,k)$ for specific values of k have been studied in several places: Stanley [10] computes $T(n,k)$ for $k \geq 7 \cdot 2^{n-4}$; Benoumhani [1] computes these values for $k \leq 12$; Kolli [6] computes $T(n,k)$ for $k \geq 3 \cdot 2^{n-3}$ and $n \geq 4$.

The results above show that $T(n,k) = 0$ for many large values of k. For a given n, it is then natural to ask: what is the smallest value of k, so that $T(n,k) = 0$?

Definition 1.1. For an integer $n \geq 1$, let $f(n) \geq 2$ be the smallest integer so that there exists no topology on n points having $f(n)$ open sets.

Equivalently, $f(n)$ is the largest number so that there exists a topology on n points with k open sets for all $2 \leq k < f(n)$. In [1], Benoumhani observes that one can easily construct a topology with k open sets for $k \in [2, 2n]$, and asks whether $T(n,k) > 0$ for all $k \in [2, 2^{n-2}]$. In other words, whether $f(n) > 2^{n-2}$ for all n. The answer to this question is negative as the following examples show.

Example 1.2. There is no topology on 9 points having 127 open sets. That is, $T(9, 127) = 0$.

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Example 1.3. There is no topology on 10 points having 191 open sets. That is, $T(10, 191) = 0$.

These results were obtained by having Stembridge's MAPLE package [12] count the order ideals in all isomorphism classes of posets with at most 10 elements. The relationship between posets and topologies is discussed in Section 2.

In this paper we go beyond simply answering Benoumhani's question: we obtain exponential lower bounds for $f(n)$, and thus a large interval of integers k for which $T(n, k) > 0$. To this end we introduce and examine the following sequence.

Definition 1.4. For an integer $k \geq 2$, let $m(k)$ be the smallest positive integer such that there exists a topology on $m(k)$ points having k open sets.

The above examples can be reformulated as: $m(127) > 9$ and $m(191) > 10$.

In Section 3 we obtain logarithmic upper bounds for $m(k)$, the main result being the following.

Theorem. *For all $k \geq 2$,*

$$m(k) \leq (4/3)\lfloor \log_2 k \rfloor + 2.$$

The proof is constructive. That is, we provide an algorithm to construct a topology with k open sets using no more than $(4/3)\lfloor \log_2 k \rfloor + 2$ points. As $f(n)$ is the smallest value of k such that $m(k) > n$ (cf. Remark 2.7), the theorem yields the following bound for $f(n)$.

Corollary. *For all $n \geq 1$,*

$$f(n) > 2^{3(n-2)/4}.$$

That is, $T(n, k) > 0$ for all $k \in [2, 2^{3(n-2)/4}]$.

Thus this paper focuses on the values $\{m(k)\}$, and finding a close upper bound for the sequence. The MAPLE program [12] can compute the initial values of this sequence, presented in Table 1 for $k \in [2, 35]$. This is sequence A137813 of [9].

k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$m(k)$	1	2	2	3	3	4	3	4	4	5	4	5	5	5	4	5	5	6

k	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35
$m(k)$	5	6	6	6	5	6	6	6	6	7	6	7	5	6	6	7

TABLE 1. The minimum number of points $m(k)$ needed to make a topology having k open sets, as computed by [12], for $k \in [2, 35]$.

The same computation also gives us the values of $f(n)$ for $n \in [1, 10]$. These values are displayed in Table 2, where they are also compared to the result of Theorem 3.11. The table indicates, as expected, that the bound is not strict. However, these data points do not contradict the possibility that $2^{3(n-2)/4}$ may give the correct growth rate for $f(n)$.

We conclude this introduction by outlining the organization of the paper. In Section 2 we recall basic definitions and describe machinery we will use throughout the proofs. This includes the correspondence between topologies and posets, under which open sets correspond to order ideals. We also develop methods to compute

n	1	2	3	4	5	6	7	8	9	10
$f(n)$	3	5	7	11	19	29	47	79	127	191
$\lfloor 2^{3(n-2)/4} \rfloor + 1$	1	2	2	3	5	9	14	23	39	65

TABLE 2. The values of $f(n)$ for $n \leq 10$. The bottom row is the size of the smallest topology not obtained by Theorem 3.11. That is, the bottom row is 1 more than the bound $2^{3(n-2)/4}$ obtained in Theorem 3.11, rounded down to the nearest integer.

the number of order ideals in a poset. In Section 3 we prove the main theorems, giving proofs of logarithmic upper bounds for $m(k)$, and consequently exponential lower bounds for $f(n)$. The proofs are constructive in that we explicitly show how to construct a topology on n points having k open sets for $k \in [2, 2^{3(n-2)/4}]$. In Section 4 we apply the constructions from Section 3 to the situation where the minimal neighborhood of each point must have at least m points, and obtain a similar interval of obtainable topology sizes. In Section 5 we discuss instances where the constructions in Section 3 are more efficient, giving topologies on fewer points than the bounds suggest. Finally, in Section 6, we make general observations about the sequences, $\{m(k)\}$ and $\{f(n)\}$, comparing them to other known sequences.

2. Machinery

We begin by recalling the definition of a topology.

Definition 2.1. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X, such that $\emptyset, X \in \mathcal{T}$, and $\mathcal{T}$ is closed under arbitrary union and finite intersection. Elements in $\mathcal{T}$ are called *open sets*. The *size* of a topology is the number of open sets. In other words, the size of the topology is the cardinality of $\mathcal{T}$.

The following class of topologies is of special importance in this article.

Definition 2.2. A T_0 *topology* on a set X is a topology on X such that, for any pair of distinct points in X, there exists an open set containing one of these points and not the other. In other words, any two points in a T_0 topology can be distinguished topologically.

In this paper we are only concerned with topologies on finite sets X. As X has only finitely many subsets, a topology on X is in fact closed under arbitrary intersection. Consequently, for a point $x \in X$, we can form the minimal open set containing x by taking the intersection

$$U_x = \bigcap_{\substack{U \in \mathcal{T} \\ x \in U}} U.$$

These minimal open sets determine $\mathcal{T}$, since

$$U = \bigcup_{x \in U} U_x$$

for all $U \in \mathcal{T}$.

For distinct x and y, minimality implies that the sets U_x and U_y are either disjoint, or one is contained in the other. Thus we can make the following definition.

Definition 2.3. For a topology $\mathcal{T}$ on a finite set X, let $P(\mathcal{T})$ be the preorder relation on X obtained by setting $x \leq y$ when $U_x \subseteq U_y$.

This assignment is a well-known bijection, as recorded in the following lemma.

Lemma 2.4. *For a finite set X, the assignment $\mathcal{T} \mapsto P(\mathcal{T})$ gives a bijective correspondence between topologies on X and preorders on X. Under this assignment, T_0 topologies correspond to partial orders.*

Proof. It is easy to reconstruct the minimal sets U_x (and hence the topology $\mathcal{T}$) from $P(\mathcal{T})$, thus obtaining an inverse for the assignment $\mathcal{T} \mapsto P(\mathcal{T})$. The last claim follows because $\mathcal{T}$ is a T_0 topology if and only if $U_x \neq U_y$ whenever $x \neq y$. □

There is a standard way to collapse a topology $\mathcal{T}$ on a set X into a T_0 topology of the same size. First, let X^0 be the set of equivalence classes formed by the relation "$x \sim y$ if $U_x = U_y$", and let $\pi\colon X \to X^0$ be the canonical projection. One then obtains a T_0 topology $\mathcal{T}^0$ on X^0 by setting

$$\mathcal{T}^0 = \{\pi(U) \mid U \in \mathcal{T}\}. \tag{1}$$

The size of $\mathcal{T}^0$ is clearly equal to the size of $\mathcal{T}$. Furthermore, $P(\mathcal{T}^0)$ is the poset obtained from the preorder $P(\mathcal{T})$ in the standard way by identifying elements x and y such that $x \leq y$ and $y \leq x$.

Example 2.5. Let $\mathcal{T}$ be the topology on $\{1, \ldots, 8\}$ with minimal open sets $U_1 = \{1\}$, $U_2 = \{2\}$, $U_3 = \{1, 2, 3\}$, $U_4 = \{1, 2, 4\}$, $U_5 = \{5\}$, $U_6 = \{1, 2, 4, 5, 6\}$, and $U_7 = U_8 = \{1, 2, 4, 5, 6, 7, 8\}$. This is not a T_0 topology because the points 7 and 8 are not distinguishable topologically. The induced T_0 topology, $\mathcal{T}^0$, is homeomorphic to the topology on $\{1, \ldots, 7\}$ with minimal open sets $U_1 = \{1\}$, $U_2 = \{2\}$, $U_3 = \{1, 2, 3\}$, $U_4 = \{1, 2, 4\}$, $U_5 = \{5\}$, $U_6 = \{1, 2, 4, 5, 6\}$, and $U_7 = \{1, 2, 4, 5, 6, 7\}$. The poset $P(\mathcal{T}^0)$ is depicted in Figure 1.

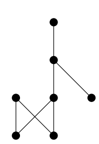


FIGURE 1. The poset $P(\mathcal{T}^0)$ corresponding to the T_0 topology $\mathcal{T}^0$ induced by the topology $\mathcal{T}$ in Example 2.5.

The following lemma describes the relationship between the sequences $m(k)$ and $f(n)$, and indicates the role of T_0 topologies.

Lemma 2.6. *Let $k \geq 2$ be an integer.*

(a) *$m(k)$ is the minimum number such that there exists a T_0 topology on $m(k)$ points having k open sets.*
(b) *If $T(n, k) > 0$ for some n, then $T(n', k) > 0$ for all $n' > n$.*

Proof. (a) A topology $\mathcal{T}$ with k open sets on a minimal number of points must be a T_0 topology, for otherwise $\mathcal{T}^0$, as defined in equation (1), is a topology with k open sets on fewer points. Thus adding the T_0 restriction does not increase the minimal number of points needed for a topology with k open sets.

(b) Suppose $\mathcal{T}$ is a topology of size k on a set X with n points. Pick a point $x \in X$ that is minimal in the preorder $P(\mathcal{T})$, and form the topology $\mathcal{T}'$ by inserting $n' - n$ additional points into U_x. Then $\mathcal{T}'$ is a topology of size k on n' points. □

Remark 2.7. From the previous lemma, it follows that $f(n)$ is the smallest integer such that $m\big(f(n)\big) > n$. We stress that the analogous statement is not true for T_0 topologies, as there is no analogue of part (b) of the lemma for T_0 topologies. Indeed, a T_0 topology on n points necessarily has at least $n+1$ open sets.

In view of the previous lemma we focus our attention on T_0 topologies and posets throughout the rest of the paper. For the remainder of this section we investigate how to calculate the size of a T_0 topology using properties of its associated poset.

Definition 2.8. An *order ideal* in a poset P is a subset $I \subseteq P$ such that if $y \in I$ and $x < y$, then $x \in I$. A *dual order ideal* in P is a subset $I \subseteq P$ such that if $x \in I$ and $x < y$, then $y \in I$.

Dual order ideals may also be called *filters*.

Definition 2.9. Let P be a poset. An *antichain* in P is a subset $A \subseteq P$ such that x and y are incomparable for all distinct $x, y \in A$.

The following facts are straightforward to prove.

Lemma 2.10. *Let $\mathcal{T}$ be a T_0 topology. The following correspondences are bijections:*

$$\{\textit{open sets in } \mathcal{T}\} \longleftrightarrow \{\textit{order ideals in } P(\mathcal{T})\} \longleftrightarrow \{\textit{antichains in } P(\mathcal{T})\}.$$

Proof. By definition of $P(\mathcal{T})$, an open set in $\mathcal{T}$ is the same as an order ideal in $P(\mathcal{T})$. An order ideal in a poset corresponds to the antichain consisting of the maximal elements in that order ideal. □

Definition 2.11. Let $j(P)$ be the number of order ideals in a poset P.

Lemma 2.10 implies that $j(P(\mathcal{T})) = |\mathcal{T}|$.

Definition 2.12. For a poset P and an element $x \in P$, let P_x be the poset obtained from P by removing all elements comparable to x. Let $P \setminus x$ be the poset obtained from P by removing only the element x.

Given a poset P, the number of order ideals in P can be computed in an iterative manner using the following lemma, which is a key tool in the proof of the main results in the paper.

Lemma 2.13. *Given a poset P and an element $x \in P$,*

$$j(P) = j(P \setminus x) + j(P_x).$$

Proof. Suppose x is an element of P, and consider an antichain $A \subseteq P$. If x is not in A, then A is an antichain in $P \setminus x$. If $x \in A$, then no element comparable to x is in A, so $A \setminus x$ is an antichain in P_x. □

Counting the number of antichains in a poset is a #P-complete problem (see [7]). This computational difficulty is the reason that the data presented in Tables 1 and 2 do not consider posets with more than 10 elements. However, the main proofs in this article build posets by inductively adding a single element at a time, and hence are undisturbed by the computational complexity.

Two elementary operations for constructing posets are the *direct sum* (also called *disjoint union*) and the *ordinal sum* of two posets.

Definition 2.14. Let P and Q be posets on the sets X and Y, respectively, with order relations R and S, respectively. The direct sum $P+Q$ is the poset defined on $X \cup Y$, with order relations $R \cup S$. The ordinal sum $P \oplus Q$ is the poset defined on $X \cup Y$, with order relations $R \cup S \cup \{x \leq y \mid x \in X, y \in Y\}$.

The number of ideals in a poset resulting from these operations can be calculated easily.

Lemma 2.15. *Let P and Q be posets. Then*

$$j(P+Q) = j(P) \cdot j(Q), \text{ and} \tag{2}$$

$$j(P \oplus Q) = j(P) + j(Q) - 1. \tag{3}$$

Proof. Consider an antichain A in the poset $P+Q$. Elements of P and Q are incomparable in this poset, so A can be written as the disjoint union $A_P \sqcup A_Q$, where A_P is an antichain of P and A_Q is antichain of Q. This proves equation (2).

Consider an antichain A in the poset $P \oplus Q$. If A includes any elements of Q, then it does not include any elements of P, yielding $j(Q)-1$ antichains (any antichain of Q except $\emptyset$). However, if $A \cap Q = \emptyset$, then A can be any antichain of P. This proves equation (3). □

Definition 2.16. Let $\bullet$ denote the poset consisting of a single element.

The following is an immediate corollary to Lemma 2.15, which is used in Proposition 3.3 to give a simple but efficient algorithm for constructing a topology with k open sets based on the base 2 expansion of k.

Corollary 2.17. *For any poset P,*

$$\begin{aligned} j(P+\bullet) &= 2j(P); \\ j(P \oplus \bullet) &= j(P)+1 \end{aligned}$$

Proof. These equalities follow directly from Lemma 2.15 because the poset $\bullet$ has two antichains: the emptyset, and the single element $\bullet$. □

Lemma 2.15 implies the following result, which provides a crude bound on the number of points needed to make a topology with a prescribed number of open sets.

Corollary 2.18. *For all $k \geq 3$,*

$$m(k) \leq \min \left\{ 1 + m(k-1), \min_{\substack{1<d<k \\ d|k}} \{m(d) + m(k/d)\} \right\}.$$

3. Exponential bounds

Here we describe three related logarithmic upper bounds for $m(k)$. In turn, these yield exponential lower bounds for $f(n)$, and consequently, exponentially large intervals of k for which $T(n,k) > 0$. The proofs of Propositions 3.3 and 3.7 and Theorem 3.11 are constructive: given an integer $k \geq 2$, a poset P having a "small" number of elements is built so that $j(P) = k$. For large values of n, Corollaries 3.4, 3.8, and 3.12 give successively larger lower bounds for $f(n)$. It may be possible to increase this bound even further, although just how much further the function $f(n)$ can be increased is still an open question.

One of the key objects in this section is the binary expansion of k.

Definition 3.1. Set $\ell = \ell(k) = \lfloor \log_2 k \rfloor$.

Definition 3.2. Given a positive integer $k = \epsilon_\ell 2^\ell + \cdots + \epsilon_1 2^1 + \epsilon_0 2^0$ where $\epsilon_i \in \{0,1\}$ and $\epsilon_\ell = 1$, let k_2 be the string $\epsilon_\ell \cdots \epsilon_1 \epsilon_0$. Each ϵ_i is a *bit*, and a bit will henceforth be written in sans-serif font as $\mathsf{0}$ or $\mathsf{1}$.

The constructions in Propositions 3.3 and 3.7 and Theorem 3.11 are similar in that they each give a blueprint for constructing a poset with k elements based on the string k_2, while trying to use as few elements as possible. Theorem 3.11 gives the best bound for $m(k)$ when $k \geq 10$. However, it is also the most complex of the three procedures. We include the other methods for three main reasons: in some cases the simpler methods are more effective, the construction in Proposition 3.3 is partly used in the proof of Theorem 3.11, and the proof of Proposition 3.7 elucidates the proof of Theorem 3.11 by motivating and explaining the ideas behind the more complicated variant.

In each construction given in this section, we read the string k_2 from left to right, building up the poset at each bit. We start with the empty poset at the first bit, and add a disjoint element to the poset for each new bit examined. At times we add maximal elements, covering selected parts of the poset, to adjust for the value of recently read bits. The difference in the constructions lies in how and when the maximal elements are added. A common aspect of each is that the disjoint elements added with the appearance of each bit form a maximal antichain of length ℓ. This observation is useful for drawing Hasse diagrams: we will draw this antichain at the lowest level, and the elements arising from the values of the bits will be positioned over it.

Proposition 3.3. *For all $k \geq 2$,*

$$m(k) \leq 2\lfloor \log_2 k \rfloor.$$

Proof. Let $k \geq 2$ be given and consider the binary expansion $k_2 = \epsilon_\ell \cdots \epsilon_1 \epsilon_0$, where $\epsilon_\ell = \mathsf{1}$. We inductively form posets $P_0, \ldots, P_\ell$ with the property that

$$\big(j(P_i)\big)_2 = \epsilon_\ell \cdots \epsilon_{\ell-i}$$

for each i. In particular, $j(P_\ell) = k$.

Let P_0 be the empty set. For each $i > 0$, consider the bit $\epsilon_{\ell-i}$, and define

$$P_i = \begin{cases} P_{i-1} + \bullet, & \text{if } \epsilon_{\ell-i} = \mathsf{0}; \\ (P_{i-1} + \bullet) \oplus \bullet, & \text{if } \epsilon_{\ell-i} = \mathsf{1}. \end{cases}$$

Using Corollary 2.17, we see that

$$j(P_i) = \begin{cases} 2j(P_{i-1}), & \text{if } \epsilon_{\ell-i} = 0; \\ 2j(P_{i-1}) + 1, & \text{if } \epsilon_{\ell-i} = 1. \end{cases}$$

Therefore $j(P_i)$ has binary expansion $\epsilon_\ell \epsilon_{\ell-1} \cdots \epsilon_{\ell-i}$.

The number of elements used in P_ℓ is $\ell + t - 1$, where t is the number of 1s in k_2. An example of the poset P_ℓ for $k = 105$ is drawn in Figure 2. We have $t \leq \ell + 1$, so $m(k) \leq 2\lfloor \log_2 k \rfloor$. □

Corollary 3.4. *For all $n \geq 1$,*

$$f(n) > 2^{n/2}.$$

That is, $T(n,k) > 0$ for all $k \in [2, 2^{n/2}]$.

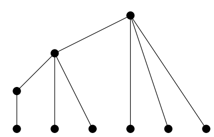


FIGURE 2. The method of Proposition 3.3 applied to $k = 105$, where $k_2 = 1101001$.

Note that $2\lfloor \log_2 105 \rfloor$ is greater than the number elements in the poset in Figure 2, but this should not be surprising given the number of 0s in 105_2. Situations where the procedures of this section may be more efficient will be discussed in Section 5.

The procedure described in the proof of Proposition 3.3 examines one bit of k_2 at a time, adding an element for each position in the string, and possibly adding another element if the bit is 1, using Corollary 2.17 to keep track of the number of ideals. Proposition 3.7 below increases the efficiency by looking at pairs of bits at a time. To do this, we first need an appropriate replacement for Corollary 2.17 to keep track of the number of ideals.

Definition 3.5. A poset is of *double type* if it contains a dual order ideal isomorphic to the poset $\bullet \oplus \bullet$.

The importance of the poset $\bullet \oplus \bullet$ is that $j(\bullet \oplus \bullet) = 3$, and it also has a dual order ideal $\bullet$ with $j(\bullet) = 2$. This allows us to adjust for the values of binary substrings 11 and 10 in k_2 by adding a single maximal element, as the following lemma shows.

Lemma 3.6. *Given a poset P of double type, and $r \in \{2, 3\}$, there is a poset P' of double type and with $j(P') = 4j(P) + r$, formed by adding three elements to the poset P.*

Proof. Add two elements to P to form the poset $Q = P + \{x_1\} + \{x_2\}$. By Corollary 2.17, we have $j(Q) = 4j(P)$.

If $r = 2$ (that is, $r_2 = 10$), form P' by adding an element y to Q, greater than everything except x_2. The subposet $\{x_1 \lessdot y\} \cong \bullet \oplus \bullet$ is a dual order ideal in P', and thus P' is of double type. Applying Lemma 2.13 (with $x = y$) implies that

$$j(P') = j(Q) + j(\{x_2\}) = 4j(P) + 2.$$

Similarly, if $r = 3$ (that is, $r_2 = 11$), form P' by adding an element y to Q, greater than everything except the dual order ideal $\bullet \oplus \bullet$ required to be in P. This $\bullet \oplus \bullet$ is still a dual order ideal in P', so P' is of double type. Furthermore, again by Lemma 2.13,

$$j(P') = j(Q) + j(\bullet \oplus \bullet) = 4j(P) + 3.$$

In each case, P' is a poset of double type with $j(P') = 4j(P) + r$, obtained by adding three elements to P. □

Proposition 3.7. *For all $k \geq 2$,*

$$m(k) \leq (3/2)\lfloor \log_2 k \rfloor + 1.$$

Proof. For a given integer $k \geq 2$, we construct a poset P with k open sets. As in the proof of Proposition 3.3, let $k_2 = \epsilon_\ell \cdots \epsilon_1 \epsilon_0$ be the binary expansion of k. We will inductively construct posets P_i for certain $i \in [0, \ell]$ with the property that $\big(j(P_i)\big)_2 = \epsilon_\ell \cdots \epsilon_{\ell-i}$. The process ends when P_ℓ is defined, and we take $P := P_\ell$.

If ϵ_ℓ is the only bit equal to 1, then set P to be a poset consisting of ℓ disjoint elements. Otherwise, let s be the smallest positive integer such that $\epsilon_{\ell-s} = 1$. Let P_s be the poset $(s \cdot \bullet) \oplus \bullet$, where $s \cdot Q$ denotes the direct sum $Q + \cdots + Q$ of s copies of the poset Q. Then $j(P_s) = 2^s + 1$, which has binary expansion $\epsilon_\ell \cdots \epsilon_{\ell-s}$. Furthermore, the poset P_s has $s + 1$ elements and is of double type.

The remainder of the proof is inductive. Assume that P_i has been defined, is of double type, and that $j(P_i)$ has binary expansion $\epsilon_\ell \cdots \epsilon_{\ell-i}$. Consider the bit $\epsilon_{\ell-(i+1)}$. If $\epsilon_{\ell-(i+1)} = 0$, then set $P_{i+1} = P_i + \bullet$. Otherwise, unless $\ell - (i + 1) = 0$, the substring $\epsilon_{\ell-(i+1)}\epsilon_{\ell-(i+2)}$ is either 11 or 10. By Lemma 3.6, we can form a poset P_{i+2} of double type such that $j(P_{i+2})$ has binary expansion $\epsilon_\ell \cdots \epsilon_{\ell-(i+2)}$, by adding three elements to P_i. If $\epsilon_{\ell-(i+1)} = 1$ and $\ell - (i + 1) = 0$, then set $P_\ell = (P_i + \bullet) \oplus \bullet$.

An example of the poset as constructed by this procedure for $k = 5550$ is depicted in Figure 3.

To construct P, we first used $s + 1$ elements to construct P_s, accounting for the leftmost $s+1$ bits in k_2. After that, we either add one element to advance one bit, or add three elements to advance two bits, until the end where two elements may need to be added for the last bit. Therefore

$$\begin{aligned} |P| &\leq (s + 1) + (\ell - s) + \lceil (\ell - s)/2 \rceil \\ &= \ell + 1 + \lceil (\ell - s)/2 \rceil \\ &\leq \ell + 1 + \lceil (\ell - 1)/2 \rceil. \end{aligned}$$

Considering cases for the parity of $\ell - 1$, one sees that

$$\ell + 1 + \lceil (\ell - 1)/2 \rceil \leq (3/2)\lfloor \log_2 k \rfloor + 1,$$

finishing the proof. □

Corollary 3.8. *For all $n \geq 1$,*

$$f(n) > 2^{2(n-1)/3}.$$

That is, $T(n, k) > 0$ for all $k \in [2, 2^{2(n-1)/3}]$.

Note that $1.5\lfloor \log_2 5550 \rfloor + 1$ is greater than the number elements in the poset in Figure 3, but, again, this should not be surprising given the number of 0s in 5550_2.

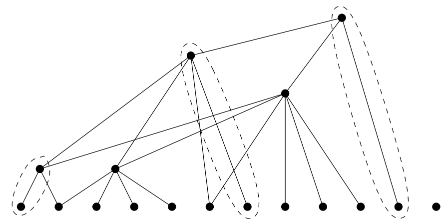
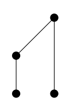
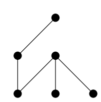
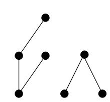


FIGURE 3. The method of Proposition 3.7 applied to $k = 5550$, where $k_2 = \mathsf{1010110101110}$. The dual order ideals isomorphic to $\bullet \oplus \bullet$ which are defined by the procedure are circled.

The bound obtained in Proposition 3.7 by considering pairs of consecutive bits in k_2 is better than the function obtained in Proposition 3.3. In fact this bound can be improved still further by considering triples of consecutive bits in k_2, as shown below, although this is significantly more complicated than the previous methods. As discussed at the end of the section, there is no analogous method for considering quadruples of consecutive bits in k_2.

Definition 3.9. A poset is of *triple type* if it contains a dual order ideal isomorphic to one of the following posets, named as indicated.

Type 1:  Type 2:  Type 3: 

The motivation for this definition is similar to that for double type. If P is isomorphic to a poset of Type 1, 2, or 3, and Q is the poset obtained by adding three disjoint points to P, then for each $r \in \{4, 5, 6, 7\}$, there is a dual order ideal I in Q with $j(I) = r$.

Lemma 3.10. *Given a poset P of triple type, and an integer $r \in \{4, 5, 6, 7\}$, there is a poset P' of triple type and with $j(P') = 8j(P) + r$, formed by adding four elements to the poset P.*

Proof. Add three elements to P to form the poset $Q = P + \{x_1\} + \{x_2\} + \{x_3\}$. By Corollary 2.17, we have $j(Q) = 8j(P)$.

Let I be a dual order ideal in P that is isomorphic to one of the posets illustrated in Definition 3.9, and let J be the dual order ideal $I + \{x_1\} + \{x_2\} + \{x_3\}$ in Q. To complete the proof, we will form a new poset P' by adding a maximal element y to Q such that the following three conditions are satisfied

- P' is of triple type,
- $y > x$ for all $x \in Q \setminus J$,
- $j(J_y) = r$, with notation as in Definition 2.12.

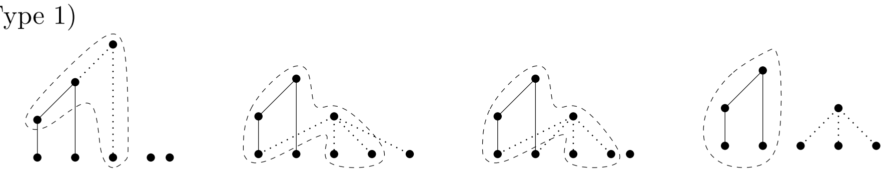
Combined the second and third condition imply that $P'_y = J_y$, and by Lemma 2.13 we have

$$j(P') = j(P' \setminus y) + j(P'_y) = j(Q) + j(J_y) = 8j(P) + r,$$

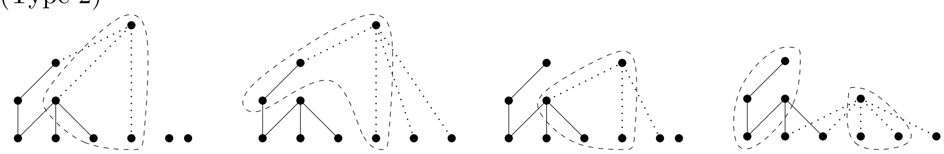
as desired.

There are twelve cases to consider for adding the element y, depending on the type of I and the value of r. The figures below show how to place y in relation to the dual order ideal J in each case. As $y > x$ for all $x \in Q \setminus J$, this shows how to add y to Q. In each case the dual order ideal I is drawn with solid lines, and the dual order ideal making P' of triple type is circled. Of the four new elements in each figure, the maximal of these is y. The first figure in each row corresponds to the case $r = j(J_y) = 4$, the second to the case $r = j(J_y) = 5$, the third to $r = j(J_y) = 6$, and the fourth to $r = j(J_y) = 7$.

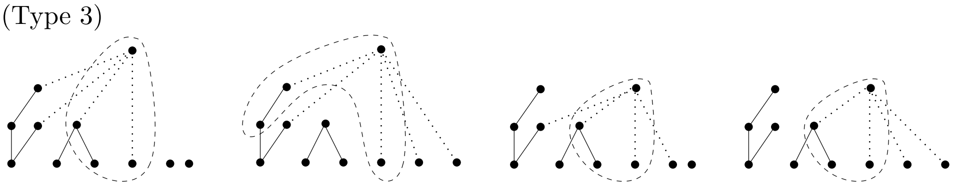
(Type 1)



(Type 2)



(Type 3)



□

Theorem 3.11. *For all $k \geq 2$,*

$$m(k) \leq (4/3)\lfloor \log_2 k \rfloor + 2.$$

Proof. The approach is similar to the proof of Proposition 3.7, and we only outline the construction. Let $k \geq 2$ be a fixed integer, and let $k_2 = \epsilon_\ell \cdots \epsilon_0$ be the binary expansion of k. We construct a poset P with $j(P) = k$. If k has fewer than three bits equal to 1 in its binary expansion, use the construction in Proposition 3.3 to obtain a poset with no more than $\ell + 1$ elements. Otherwise, let s be such that $\epsilon_{\ell-s}$ is the third nonzero bit from the left in k_2. Using the construction in Proposition 3.3 we obtain a poset P_s with $s + 2$ elements such that $j(P_s)$ has binary expansion $\epsilon_\ell \cdots \epsilon_{\ell-s}$. Observe that P_s is of triple type as it contains a dual order ideal isomorphic to Type 1 in Definition 3.9.

As in the proof of Proposition 3.7, we now move rightward in the binary expansion of k. If we encounter the bit 0, we add a single disjoint point to our poset and move on. If we encounter the bit 1, we consider this bit and the two immediately following it. They form one of the subsequences 100, 101, 110 or 111. In each case the corresponding integer belongs to the set $\{4, 5, 6, 7\}$, and we can apply Lemma 3.10 to obtain a new poset of triple type incorporating the three bits under scrutiny, by adding four elements. Finally, when there are $i < 3$ bits left we can incorporate them into the poset by adding $i+1$ points, using Corollary 2.17 if $i = 1$ and Lemma 3.6 if $i = 2$.

A counting argument similar to the one in Proposition 3.7 shows that

$$\begin{aligned} |P| &\leq (s+2) + (\ell - s) + \lceil (\ell - s)/3 \rceil \\ &= \ell + 2 + \lceil (\ell - s)/3 \rceil \\ &\leq \ell + 2 + \lceil (\ell - 2)/3 \rceil. \end{aligned}$$

Examination of cases based on the remainder $\ell \bmod 3$ gives that

$$\ell + 2 + \lceil (\ell - 2)/3 \rceil \leq (4/3)\lfloor \log_2 k \rfloor + 2,$$

finishing the proof. □

Corollary 3.12. *For all* $n \geq 1$,

$$f(n) > 2^{3(n-2)/4}.$$

That is, $T(n,k) > 0$ *for all* $k \in [2, 2^{3(n-2)/4}]$.

The successive results in Propositions 3.3 and 3.7 and Theorem 3.11 suggest that even better bounds might be obtained by adapting the constructions to consider four bits of k_2 at a time, for any $k \geq 2$. However, this is not possible with our approach, as the following argument shows. Suppose that k_2 has at least four 1 bits. If the fourth 1 bit from the left is $\epsilon_{\ell-s}$, then the first step would be to use the procedure in Proposition 3.3 to construct a poset P_s on $s+3$ elements so that $j(P_s)_2 = \epsilon_\ell \cdots \epsilon_{\ell-s}$. However, no dual order ideal (nor order ideal, for that matter) in $P_s + 4 \cdot \bullet$ has 13 antichains, so if the leftmost $s+5$ digits of k_2 are $\epsilon_\ell \cdots \epsilon_{\ell-s}1101$, then there is no way to add a single maximal element to $P_s + 4 \cdot \bullet$ to obtain a poset having $16j(P_s) + 13$ antichains. Therefore the result of Theorem 3.11 is the best that can be obtained with this technique.

4. Specified minimal set sizes

The results in the previous section can be generalized by looking at topologies where the minimal open sets $\{U_x\}$ have specified sizes. An extremal case of this, related to cardinalities of distributive lattices with a specified number of join-irreducibles of each rank, is treated in [11]. One version of this generalization is very easy to handle by modifying the construction described in Theorem 3.11 to produce topologies with specified minimal set sizes.

Definition 4.1. Let $T_m(n,k)$ be the number of topologies on n points having k open sets, where the smallest neighborhood of each point has at least m elements.

Proposition 4.2. $T_m(n,k) > 0$ *for all* $n \geq m$, $m \geq 1$, *and* $k \in [2, 2^{\frac{3(n-2)}{3m+1}}]$.

Proof. In a topology $\mathcal{T}$, the smallest neighborhood of a point x is the set U_x. The sets U_x with fewest elements are those where x is minimal in the preorder $P(\mathcal{T})$.

In the procedure described in the proof of Theorem 3.11, the minimal elements of the poset form an antichain of size ℓ, corresponding to each bit after ϵ_ℓ in the step-by-step reading of k_2. Therefore, requiring the smallest neighborhood of each point in $\mathcal{T}$ to contain at least m points simply means replacing each of these ℓ elements by a set of cardinality at least m. Thus, to make such a topology with k open sets, a similar argument to that in the proof of the theorem shows that one needs at most $m\ell + 2 + \lceil (\ell-2)/3 \rceil$ elements. As in the proof of the theorem,

$$m\ell + 2 + \lceil (\ell - 2)/3 \rceil \leq (m + 1/3)\ell + 2,$$

and the result follows. □

5. Better efficiency

The main result of this paper, Theorem 3.11, gives a procedure to construct a topology having k open sets, needing one extra point in the topology for each triple of bits after the first three 1s in the binary expansion k_2. There may be some situations where this procedure requires fewer points than the bounds suggest, and we highlight a few of these here.

First of all, if the binary expression k_2 includes many 0s, then there may be large portions of this expression that get skipped over by the procedure, and thus fewer triples contribute an element to the poset.

Another way to increase the efficiency of this type of procedure would be to note patterns of consecutive digits in the string k_2. For example, suppose that $k = 2^{2^r} - 1$. Thus $\ell = 2^r - 1$ and the binary string k_2 consists of 2^r repeated 1s. Then one can parse the string k_2 as

$$1 \mid 1 \mid 11 \mid 1111 \mid 11111111 \mid \ldots,$$

where each section is identical to the union of all sections to the left. Thus a new section with 2^s 1s can be handled by finding a dual order ideal in the poset with $2^{2^s} - 1$ antichains, similarly to the procedure in the proof of Theorem 3.11. An example of this for $2^{2^4} - 1$ is depicted in Figure 4.

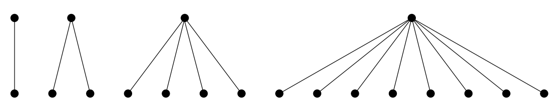


FIGURE 4. An efficient way to draw a poset with 65535 antichains, using 19 elements.

As suggested by Figure 4 and Lemma 2.15, if the number of open sets desired factors conveniently well, this may also reduce the number of points needed in the topology.

Fix positive integers a and b. If the desired number of open sets is

$$k = 1 + 2^a + 2^{2a} + \cdots + 2^{ba},$$

then the procedure in Proposition 3.3 gives a poset having $(a+1)b$ elements and k antichains. Figure 5 depicts such a poset when $a = 3$ and $b = 4$ (that is, $k = 4681$).

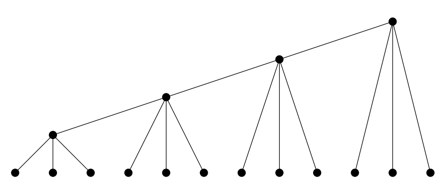


FIGURE 5. The procedure in Proposition 3.3 applied to $k = 4681$.

Now consider an integer of the form $k = x(1 + 2^a + 2^{2a} + \cdots + 2^{ba})$, where $\ell(x) + 1 \leq a$. The binary expansion of k consists of $b+1$ repeated instances of the binary expansion of x. In this situation, due to Lemma 2.15, there exists a poset having k antichains and at most

$$(a+1)b + (4/3)\lfloor \log_2 x \rfloor + 2$$

elements. Thus, integers k with repeated patterns in their binary expansion can be handled very efficiently.

6. Comparison to other sequences

Using Stembridge's MAPLE program [12], we have calculated the initial values of the sequence $\{m(k)\}$, and these have been entered into [9] as entry A137813. The terms of $m(k)$ are very similar to sequence A003313 of [9], giving the length of a shortest addition chain for an integer.

Definition 6.1. An *addition chain* for k is sequence of integers $x_0, x_1, \cdots, x_n$ such that $x_0 = 1$, $x_n = k$, and each term in the sequence is the sum of two (not necessarily distinct) numbers appearing earlier in the sequence. The *length* of the addition chain $x_0, x_1, \cdots, x_n$ is n.

For more information, both historical and mathematical, about addition chains, see [5]. Sequence A003313 of [9] is defined as follows.

Definition 6.2. For a positive integer k, let $a(k)$ be the length of the shortest possible addition chain for k.

Interestingly, the sequences $a(k)$ and $m(k)$ agree in their first 100 terms, except for $k = 71$, where $m(71) = 8$, while $a(71) = 9$. It is tempting to wonder whether $a(k)$ is an upper bound for $m(k)$. Examples suggest that "short" addition chains can be realized by posets, but this does not seem to be true for "long" addition chains. The division between "short" and "long" chains is unclear, but seems to lie above the range of values for which it is currently feasible to calculate $m(k)$. It would be interesting to compare the behavior of these sequences over a greater range.

A concrete relationship between the sequences $m(k)$ and $a(k)$ is a common upper bound.

Definition 6.3. For a positive integer k, let $b(k)$ be the length of the shortest possible addition chain for k obtained by using only the methods of factoring and binary expansion.

This is sequence A117498 in [9]. By definition, $b(k)$ is an upper bound for $a(k)$. Also from the definition it follows that $b(k)$ satisfies the inductive equation

$$b(k) = \begin{cases} 1, & \text{if } k = 2; \\ \min\left\{ 1 + b(k-1), \min\limits_{\substack{1<d<k \\ d|k}} \{b(d) + b(k/d)\} \right\}, & \text{if } k > 2. \end{cases}$$

It follows from Corollary 2.18 that $b(k)$ is an upper bound for $m(k)$. The first term where the sequences differ is $k = 23$, where $b(23) = 7$, while $a(23) = m(23) = 6$.

The sequence $f(n)$ has also been entered into [9], as sequence number A137814. The initial terms of this sequence are 3, 5, 7, 11, 19, 29, 47, 79, 127, and 191. That these are all prime numbers is not surprising: for a composite number k, Lemma 2.15 implies that one can efficiently construct a poset with k order ideals as a direct sum of two posets. Given the relationship between $m(k)$ and $a(k)$ above, it is expected that $\{f(n)\}$ be similar to sequence A003064 of [9], giving the smallest number with addition chains of length n.

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