# Encoding fusion data in the double Burnside ring

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Fusion systems model the $p$-local structure of a finite group.

Let $G$ be a finite group with Sylow subgroup $S$

**Definition**

The fusion system of $G$ on $S$ is the category $\mathcal{F} = \mathcal{F}_S(G)$ with:
-Objects: Subgroups of $S$.
-Morphisms: $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Hom}_G(P, Q)$

Here, $\mathrm{Hom}_G(P, Q)$ is the set of homomorphisms $\varphi\colon P \to Q$ that are induced by conjugation in $G$.

More generally:

**Definition**

A fusion system on a finite $p$-group $S$ is a category $\mathcal{F}$ with:

- Objects are the subgroups of $S$.
- Morphisms satisfy

$$\mathrm{Hom}_S(P,Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P,Q) \subseteq \mathrm{Inj}(P,Q),$$

  and every morphism can be factored as an isomorphism in $\mathcal{F}$ followed by a group inclusion.

**Definition (Puig)**

A fusion system is saturated if it satisfies two additional axioms, playing the role of Sylow theorems.

I “prime to $p$ axiom”

II “Maximal extension axiom”

Fusion systems of groups are saturated.

Saturated fusion systems also come up in:

- Block theory, induced by conjugation among Brauer subpairs.
- Topology, as Chevalley groups of $p$-compact groups.

Let $\mathcal{F}$ be a fusion system on $S$.

## Definition

- $P, Q \leq S$ are $\mathcal{F}$-conjugate if they are isomorphic in $\mathcal{F}$.
- $P \leq S$ is fully $\mathcal{F}$-centralized if $|C_S(P)| \geq |C_S(Q)|$ for every $Q$ that is $\mathcal{F}$-conjugate to $P$.
- $P \leq S$ is fully $\mathcal{F}$-normalized if $|N_S(P)| \geq |N_S(Q)|$ for every $Q$ that is $\mathcal{F}$-conjugate to $P$.

## Definition (Saturation Axiom I)

$\mathcal{F}$ satisfies Axiom I if the following holds for every $P \leq S$:
If $P$ is fully $\mathcal{F}$-normalized, then $P$ is fully $\mathcal{F}$-centralized and $p \nmid [\mathrm{Aut}_{\mathcal{F}}(P) : \mathrm{Aut}_S(P)]$.

This axiom replaces "$p \nmid [G : S]$".

Axiom II:

**Definition**

For $P \leq S$, and a monomorphism $\varphi\colon P \to S$, set

$$N_\varphi = \{x \in N_S(P) \mid \varphi \circ c_x \circ \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}$$

$N_\varphi$ is the largest subgroup of $N_S(X)$ to which we could hope to extend $\varphi$. ($\varphi \circ c_x \circ \varphi^{-1} = c_{\varphi(x)}$)

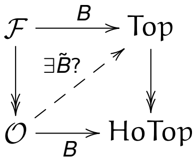
**Definition (Saturation Axiom II)**

$\mathcal{F}$ satisfies Axiom II if the following holds for every morphism $\varphi\colon P \to S$ in $\mathcal{F}$:
If $\varphi(P)$ is fully $\mathcal{F}$-centralized, then there exists a morphism $\bar{\varphi} \in \mathrm{Hom}_\mathcal{F}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

This axiom replaces "all Sylow subgroups are conjugate".

A *p-local finite group* is a saturated fusion system equipped with a classifying space.
Motivation: $BG_p^\wedge$ is a classifying space for $\mathcal{F}_S(G)$.
Have classifying space functor $B\colon \mathcal{F} \to \mathrm{Top}$.
Need to quotient out action of inner homomorphisms before taking homotopy colimit.
The orbit category $\mathcal{O}$ has same objects as $\mathcal{F}$ and morphisms

$$\mathrm{mor}_{\mathcal{O}}(P, Q) = Q \backslash \mathrm{Hom}_{\mathcal{F}}(P, Q)$$




Dwyer–Kan obstruction theory to existence and uniqueness of homotopy lifting $\tilde{B}$.
If $\tilde{B}$ exists, obtain a classifying space $\underrightarrow{\mathrm{Holim}}_{\mathcal{O}(\mathcal{F})} \tilde{B}$

Algebraic version:

### Definition

A group $P \leq S$ is $\mathcal{F}$-centric if $C_S(Q) = Z(Q)$ for all $Q$ that are $\mathcal{F}$-conjugate to $P$.
Let $\mathcal{F}^c \subseteq \mathcal{F}$ be the full subcategory of $\mathcal{F}$-centric subgroups.

### Definition

A centric linking system associated to $\mathcal{F}$ is a category $\mathcal{L}$ where

- Objects are the $\mathcal{F}$-centric subgroups
- $Z(P)$ acts freely on $\mathrm{mor}_{\mathcal{L}}(P, Q)$ with quotient $\mathrm{Hom}_{\mathcal{F}}(P, Q)$.
- $+$ technical conditions.

Think of $\mathcal{L}$ as a “crossed module extension” of $\mathcal{F}^c$ by $Z(-)$.
Corresponding obstruction theory recovers Dwyer–Kan obstructions.

Classifying space: $|\mathcal{L}|_p^\wedge$.

# Definition (BLO)

Let $\mathcal{F}$ be a fusion system over the $p$-group $S$. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$, whose objects are the $\mathcal{F}$-centric subgroups of $S$, together with a functor

$$\pi : \mathcal{L} \to \mathcal{F}^c,$$

and distinguished monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}\, P\mathcal{L}$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions.

(A) The functor $\pi$ is the identity on objects and surjective on morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, the center $Z(P)$ acts freely on $\mathrm{mor}_{\mathcal{L}}(P, Q)$ by composition (upon identifying $Z(P)$ with $\delta_P(Z(P)) \leq \mathrm{Aut}_{\mathcal{L}}(P)$), and $\pi$ induces a bijection

$$\mathrm{mor}_{\mathcal{L}}(P, Q)/Z(P) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \leq S$ and each $g \in P$, $\pi$ sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \downarrow{\scriptstyle \delta_P(g)} & & \downarrow{\scriptstyle \delta_Q(\pi(f)(g))} \\ P & \xrightarrow{f} & Q. \end{array}$$

Goal in theory of $p$-local finite groups:
Obtain functorial assignment of unique classifying space to each saturated fusion system.
Results:

- Existence for $rk(P) < p^3$
- Uniqueness for $rk(P) < p^2$
- Existence and uniqueness in group case
- Functoriality: ???

Works nicely in stable homotopy:

**Theorem (KR)**

*Have functorial assignment of classifying spectra to saturated fusion systems.*

## The double Burnside ring

**Definition**

A $(G_1, G_2)$-biset is a set with a right $G_1$-action and a commuting, free left $G_2$-action.

The isomorphism classes of finite $(G_1, G_2)$-bisets form a monoid under disjoint union.

**Definition**

The Burnside module $A(G_1, G_2)$ is the group completion of this monoid.

An element of $A(G_1, G_2)$ is a formal difference $[X] - [Y]$ of isomorphism classes of finite $(G_1, G_2)$-bisets.

Basis for $A(G_1, G_2)$:

A $(G_1, G_2)$-pair is a pair $(H, \varphi)$, where

$$H \leq G_1, \varphi\colon H \to G_2.$$

Conjugacy: $(H_1, \varphi_1) \sim (H_2, \varphi_2)$ if $\exists g_1 \in G_1$, $\exists g_2 \in G_2$ s.t.

$$\begin{array}{ccc} H_1 & \xrightarrow{\varphi_1} & \varphi_1(H_1) \\ \cong \big\downarrow c_{g_1} & & \cong \big\downarrow c_{g_2} \\ H_2 & \xrightarrow{\varphi_2} & \varphi_2(H_2). \end{array}$$

Write $[H, \varphi]_{G_1}^{G_2}$ (or just $[H, \varphi]$) for the conjugacy class of $(H, \varphi)$.

$A(G_1, G_2)$ is a free $\mathbb{Z}$-module with basis indexed by conjugacy classes of $(G_1, G_2)$-pairs.

The basis element $[H, \varphi]_{G_1}^{G_2}$ corresponds to the biset

$$(G_1 \times G_2)/\Delta_H^{\varphi},$$

where

$$\Delta_H^{\varphi} = \{(h, \varphi(h) \mid h \in H\},$$

and actions are given by

$$b(x, y)a = (a^{-1}x, by),$$

for $a, x \in G_1$ and $b, y \in G_2$.

## Definition

The Burnside category $\underline{A}$ is the category with
-Objects: Finite groups
-Morphisms: $\text{mor}_{\underline{A}}(G_1, G_2) := A(G_1, G_2)$
-Composition:

$$A(G_2, G_3) \times A(G_1, G_2) \longrightarrow A(G_1, G_3)$$

$$(\Omega', \Omega) \mapsto \Omega' \circ \Omega := \Omega' \times_{G_2} \Omega$$

This can be described on basis elements by the double coset formula:

$$[K, \psi]_{G_2}^{G_3} \circ [H, \varphi]_{G_1}^{G_2} = \sum_{x \in K \backslash G_2 / \varphi(H)} \left[\varphi^{-1}\left(\varphi\left(H\right) \cap K^x\right), \psi \circ c_x \circ \varphi\right]_{G_1}^{G_3}$$

In particular, $A(G, G)$ is a ring, called the double Burnside ring of $G$.

When $S \leq G$ is Sylow, the $(S, S)$-biset $[G]$ plays a special role. Linckelmann–Webb formalized this for fusion systems.

**Definition (Linckelmann–Webb)**

A characteristic element for $\mathcal{F}$ is an element $\Omega \in A(S, S)_{(p)}$ such that

a) $|\Omega/S|$ is prime to $p$

b) For all $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$,
$\Omega \circ [P, \varphi]_P^S = \Omega \circ [P, \mathrm{incl}]_P^S$ (right $\mathcal{F}$-stable),
and
$[\varphi(P), \varphi^{-1}]_S^P \circ \Omega = [P, \mathrm{id}]_S^P \circ \Omega$ (left $\mathcal{F}$-stable).

c) $\Omega$ lies in the span of $\{[P, \varphi] \mid P \leq S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S).\}$

Motivation:

b) $xG = G = Gx$ for $x \in G$.

c) $$[G]_S^S = \sum_{x \in S\backslash G/S} [S \cap S^x, c_x]_S^S$$

## Theorem (Broto–Levi–Oliver)

*Every saturated fusion system has a characteristic biset.*

In cohomology with $\mathbb{F}_p$-coefficients, $\Omega$ induces an idempotent with image the $\mathcal{F}$-stable elements in $H^*(S; \mathbb{F}_p)$.
This generalizes the transfer $H^*(S) \to H^*(G)$.

What about other Mackey functors $M$?
-$M([\Omega])$ generally not idempotent
-$[\Omega]$ is not unique

## Theorem (KR)

*Every saturated fusion system $\mathcal{F}$ has a unique characteristic idempotent $\omega_{\mathcal{F}}$.*

**Proof.**

- If $\Omega$ characteristic element, then $\Omega^n$ is also one.
- Some power of $\Omega$ is idempotent $(\mathrm{mod}\, p)$ (since $\mathbb{F}_p \otimes A(S,S)$ is finite).
- Take characteristic biset $\Omega$ that is idempotent $(\mathrm{mod}\, p)$. Then $\Omega^{p^n}$ is idempotent $(\mathrm{mod}\, p)$.
- Conclude that $\Omega, \Omega^p, \Omega^{p^2}, \ldots$ is a Cauchy sequence converging to an idempotent $\omega$ in $A(S,S)_p^\wedge$.
- Hard part: Coefficients in basis decomposition of $\omega$ satisfy fully determined system of equations, giving uniqueness.
- Since equations have integer coefficients, $\omega$ lies in $A(S,S)_{(p)}$.

□

The "hard part" involves describing $\omega_{\mathcal{F}} A(S,S)^{\wedge}_{p} \omega_{\mathcal{F}}$. Basically, multiplying by $\omega$ "quotients out" $\mathcal{F}$-conjugacy.
This has another important consequence.

**Definition**

For $X \in A(S,S)$, the stabilizer fusion system of $X$ is the fusion system $\mathrm{Stab}(X)$ on $S$ with morphism sets

$$\{\varphi \in \mathrm{Inj}(P,Q) \mid X \circ [P,\varphi]_P^S = X \circ [P,\mathrm{incl}]_P^S\}$$

**Corollary (KR)**

*If $\Omega$ is a characteristic biset (or idempotent) for $\mathcal{F}$, then $\mathrm{Stab}(\Omega) = \mathcal{F}$.*

This has an interesting interpretation in stable homotopy: We can recover $\mathcal{F}_S(G)$ from the stable homotopy type of the map $BS \to BG^{\wedge}_p$, but not from the stable homotopy type of $BG^{\wedge}_p$ [Martino–Priddy].

Saturation can also be detected in the Burnside ring.

**Theorem (Puig,KR–Stancu)**

*Let $\mathcal{F}$ be a fusion system on $S$. If $\mathcal{F}$ has a characteristic biset (or idempotent), then $\mathcal{F}$ is saturated.*

This is a first radically different formulation of saturation.

The proof goes by counting fixed points.

For each $[P, \varphi]$, we have a fixed-point homomorphism

$$\Phi_{[P,\varphi]} \colon A(S, S) \to \mathbb{Z}, \; X \to |X^{\Delta_P^{\varphi}}|.$$

By Burnside, this gives an injection

$$\Phi \colon A(S, S) \xrightarrow{\prod_{[P,\varphi]} \Phi_{[P,\varphi]}} \prod_{[P,\varphi]} \mathbb{Z}.$$

Condition c) becomes

$$\Phi_{[P,\varphi]}(\Omega) = 0$$

when $\varphi \notin \mathcal{F}$.
Condition b) becomes

$$\Phi_{[P,\varphi]}(\Omega) = \Phi_{[\varphi(P),\text{incl}]}(\Omega)$$

$$\Phi_{[P,\varphi]}(\Omega) = \Phi_{[P,\text{incl}]}(\Omega)$$

when $\varphi \in \mathcal{F}$.

Looking at

$$|(S\backslash\Omega)^P| \equiv |S\backslash\Omega| \not\equiv 0 \pmod p,$$

we get

$$\sum_{[\varphi]\in S\backslash \operatorname{Hom}_{\mathcal{F}}(P,S)} \frac{\Phi_{[P,\varphi]}(\Omega)}{|C_S(\varphi(P))|} \not\equiv 0 \pmod p,$$

where $m = \Phi_{[P,\varphi]}(\Omega)$ is constant. (Condition b))

We deduce that $\varphi(P)$ is fully $\mathcal{F}$-centralized if and only if

$$\frac{m}{|C_S(\varphi(P))|} \not\equiv 0 \pmod p.$$

Similarly we obtain

$$\sum_{[Q]_S \in [P]_{\mathcal{F}}} \frac{m \cdot |\operatorname{Aut}_{\mathcal{F}}(Q)|}{|N_S(Q)|} \not\equiv 0 \pmod{p},$$

and deduce that

$$Q \text{ is fully } \mathcal{F}\text{-normalized}$$

$$\Updownarrow$$

$$\frac{m \cdot |\operatorname{Aut}_{\mathcal{F}}(Q)|}{|N_S(Q)|} \not\equiv 0 \pmod{p}$$

$$\Updownarrow$$

$$\frac{m}{|C_S(Q)|} \cdot \frac{|\operatorname{Aut}_{\mathcal{F}}(Q)|}{|N_S(Q)|} \not\equiv 0 \pmod{p}$$

$$\Updownarrow$$

$Q$ is fully centralized and $\operatorname{Aut}_S(P)$ is Sylow in $\operatorname{Aut}_{\mathcal{F}}(P)$.

This proves Axiom I! Axiom II is similar but more complicated.

**Frobenius reciprocity:**
Let $\Delta\colon S \to S \times S$ be the diagonal.

For $(S, S)$-bisets $X$ and $Y$, let $(X \times Y) \circ \Delta$ be the set $(X \times Y)$ regarded as an $(S, S \times S)$-biset via

$$(a_1, a_2)(x, y)b = (a_1 xb, a_2 yb)$$

for $(a_1, a_2) \in S \times S, (x, y) \in X \times Y, b \in S$.

($\Delta$ is short for $[S, \Delta]_S^{S \times S} \in A(S, S \times S)$)

**Theorem (KR–Stancu)**

*If $\Omega \in A(S, S)_{(p)}$ satisfies the Frobenius reciprocity relation*

$$(\Omega \times \Omega) \circ \Delta = (\Omega \times 1) \circ \Delta \circ \Omega,$$

*then* $\mathrm{Stab}(\Omega)$ *is saturated, and $\Omega$ is a characteristic biset for* $\mathrm{Stab}(\Omega)$.

Why Frobenius reciprocity? Think of group case.

Under the Segal conjecture, the characteristic idempotent of $\mathcal{F} = \mathcal{F}_S(G)$ corresponds to the composite

$$\Sigma^\infty_+ BS \xrightarrow{B\iota} \Sigma^\infty_+ BG \xrightarrow{t} \Sigma^\infty_+ BS,$$

where $t$ is a "normalized transfer" (so $B\iota \circ t \simeq 1$).

The Frobenius reciprocity relation

$$(\omega \times \omega) \circ \Delta = (\omega \times 1) \circ \Delta \circ \omega$$

is equivalent to

$$(B\iota \wedge 1_{BS}) \circ \Delta_{BS} \circ t \simeq (1_{BG} \wedge t) \circ \Delta_{BG}.$$

On cohomology this induces

$$\mathrm{Tr}(\mathrm{Res}(x) y) = x\,\mathrm{Tr}(y).$$

## Proof.

- Need to show: If $[P, \varphi]$ appears in $\Omega$, then, for all $\psi$

$$\Phi_{[P,\psi]}(\Omega) = \Phi_{[\varphi(P),\psi\circ\varphi^{-1}]}(\Omega).$$

- Frobenius reciprocity implies that for all $\psi$ and $\varphi$

$$\Phi_{[P,\psi]}(\Omega)\Phi_{[P,\varphi]}(\Omega) = \Phi_{[\varphi(P),\psi\circ\varphi^{-1}]}(\Omega)\Phi_{[P,\varphi]}(\Omega).$$

- Suffices to show that $\Phi_{[P,\varphi]}(\Omega) \neq 0$ if $[P, \varphi]$ appears in $\Omega$.
- The fusion system generated by $\varphi$ that appear in $\Omega$ is equal to the closure of the "pre-fusion system" $\mathtt{Pre\text{-}Fix}(\Omega)$ consisting of maps $\varphi$ with $\Phi_{[P,\varphi]}(\Omega) \neq 0$.
- Enough to show that $\mathtt{Pre\text{-}Fix}(\Omega)$ is a fusion system.
- Closure under composition of isomorphisms and inverses easy. Closure under restriction hard.

□

## Corollary (KR–Stancu)

*For a finite group S, there is a bijection*

$$\{\textit{Saturated fusion systems over } S\}$$
$$\updownarrow$$
$$\{\textit{Frobenius idempotents in } \mathbb{Z}_{(p)} \otimes A(S,S)\}$$

The bijection sends a fusion system to its characteristic idempotent and a Frobenius idempotent to its stabilizer fusion system.

This gives us a completely new way to think about saturated fusion systems!

But wait, there's more!

## Application to stable splittings

For $S \leq G$ Sylow, a transfer argument shows that $BG$ is a stable summand of $BS$.

By the Segal conjecture, stable summands of $BS$ correspond to idempotents in $A(S,S)^{\wedge}_{p}$.

Martino–Priddy worked out complete stable splitting of $BS$ and asked.

**Question:** Which idempotents in $A(S,S)^{\wedge}_{p}$ correspond to classifying spaces of groups?

To answer this question we must extend the framework to saturated fusion systems.

**Answer:** An idempotent in $A(S,S)^{\wedge}_{p}$ corresponds to the classifying spectrum of a saturated fusion system if and only if it satisfies Frobenius reciprocity.

## Relation to the Adams–Wilkerson theorem

**Theorem (Adams–Wilkerson (variant))**

*Let $V$ be an elementary abelian $p$-group and let $R^* \subseteq H^* = H^*(V;\mathbb{F}_p)$ be a subring. Then $R^* = (H^*)^W$ for a subgroup $W \leq \mathrm{Aut}(S)$ of order prime to $p$ if and only if $R^* \hookrightarrow H^*$ is the inclusion of a direct summand of $R^*$-modules.*

Can generalize this to arbitrary $p$-groups, lifting from cohomology to stable homotopy. Instead of looking for rings of invariants, we look for stable elements with respects to a fusion system.

**Theorem (in progress)**

*For a finite $p$-group $S$ and $R \subseteq A$, there is a saturated fusion system $\mathcal{F}$ over $S$ such that $R$ is the ring of $\mathcal{F}$-stable elements in $A$ if and only $R \hookrightarrow A$ is the inclusion of a direct summand of $R$-modules.*

**Retractive transfer triples and $p$-local finite groups**

A retractive transfer triple over $S$ is a triple $(f, t, X)$ where

- $X$ is a $p$-complete space of finite type.
- $f\colon BS \to X$ is a homotopy monomorphism at $p$.
- $t\colon \Sigma^\infty_+ X \to \Sigma^\infty_+ BS$ is a stable retract of $f$ such that

$$(\Sigma^\infty_+ f \wedge 1_{\Sigma^\infty_+ X}) \circ \Delta_X \circ t \simeq (1 \wedge t) \circ \Delta_{BS}$$

$X$ plays the role of $BG^\wedge_p$ or $|\mathcal{L}|^\wedge_p$.
$f$ is a natural inclusion.
$t$ is a normalized transfer. $(\Sigma^\infty_+ f \circ t \simeq 1_{\Sigma^\infty_+ X})$

Haynes Miller asked the following.
**Question:** Do retractive transfer triples give a theory equivalent to *p*-local finite groups?

**Partial answer:** Every *p*-lfg gives rise to a RTT.
A RTT over an elementary abelian *p*-group is a *p*-lfg.
(This was my thesis)

**Theorem (KR)**

*A RTT $(f, t, X)$ over any $S$ gives rise to a saturated fusion over $S$.*

**Proof.**

$\omega = t \circ \Sigma^{\infty}_{+} f$ is a Frobenius idempotent. □

Remains: Relate $X$ to classifying space of $\mathtt{Stab}(\omega)$.
Use Wojtkowiak's obstruction theory.