Encoding fusion data in the double Burnside ring

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MSRI Postdoctoral Seminar
April 11, 2008

Definition

A (G_1, G_2)-biset is a set with a right G_1-action and a commuting, free left G_2-action.

The isomorphism classes of finite (G_1, G_2)-bisets form a monoid under disjoint union.

Definition

The Burnside module $A(G_1, G_2)$ is the group completion of this monoid.

An element of $A(G_1, G_2)$ is a formal difference $[X] - [Y]$ of isomorphism classes of finite (G_1, G_2)-bisets.

Basis for $A(G_1, G_2)$:

A (G_1, G_2)-pair is a pair (H, φ), where

$$H \leq G_1, \varphi \colon H \to G_2.$$

Conjugacy: $(H_1, \varphi_1) \sim (H_2, \varphi_2)$ if $\exists g_1 \in G_1$, $\exists g_2 \in G_2$ s.t.

$$\begin{array}{ccc} H_1 & \xrightarrow{\varphi_1} & \varphi_1(H_1) \\ \cong \Big\downarrow c_g & & \cong \Big\downarrow c_{g'} \\ H_2 & \xrightarrow{\varphi_2} & \varphi_2(H_2). \end{array}$$

Write $[H, \varphi]_{G_1}^{G_2}$ (or just $[H, \varphi]$) for the conjugacy class of (H, φ).

$A(G_1, G_2)$ is a free $\mathbb{Z}$-module with basis indexed by conjugacy classes of (G_1, G_2)-pairs.

The basis element $[H, \varphi]_{G_1}^{G_2}$ corresponds to the biset

$$G_1 \times G_2 / \Delta_H^{\varphi},$$

where

$$\Delta_H^{\varphi} = \{(h, \varphi(h)) \mid h \in H\},$$

and actions are given by

$$b(x, y)a = (a^{-1}x, by),$$

for $a, x \in G_1$ and $b, y \in G_2$.

Definition

The Burnside category $\underline{A}$ is the category with
-Objects: Finite groups
-Morphisms: Burnside modules $A(G_1, G_2)$
-Composition:

$$A(G_2, G_3) \times A(G_1, G_2) \longrightarrow A(G_1, G_3)$$

$$(\Omega', \Omega) \mapsto \Omega' \circ \Omega := \Omega' \times_{G_2} \Omega$$

This can be described on basis elements by the double coset formula:

$$[K, \psi]_{G_2}^{G_3} \circ [H, \varphi]_{G_1}^{G_2} = \sum_{x \in K \backslash G_2' / \varphi(H)} \left[\varphi^{-1}\left(\varphi\left(H\right) \cap K^x\right), \psi \circ c_x \circ \varphi\right]_{G_1}^{G_3}$$

In particular, $A(G, G)$ is a ring, called the double Burnside ring of G.

Let R be a ring. An R-valued global Mackey functor is a contravariant functor

$$\underline{A} \longrightarrow R - \mathrm{mod}\,.$$

This is a functor M defined on finite groups which allows
-restriction along a group homomorphism

$$\varphi : H \to G_2 \rightsquigarrow \varphi^* : M(G_2) \to M(H)$$

-transfer along a subgroup inclusion

$$H \leq G_1 \rightsquigarrow tr : M(H) \to M(G_1).$$

We can think of the biset $[H, \varphi]$ as the composite of a transfer and restriction.

Examples: Group cohomology, representation rings, ...

Let S be a Sylow subgroup of G. We have a commutative diagram

$$\begin{array}{ccc} & M(G) & \\ {}^{tr}\nearrow & & \searrow^{res} \\ M(S) & \xrightarrow{M([G]_S^S)} & M(S) \end{array}$$

where $[G]_S^S$ is G regarded as a (S, S)-biset.

Theorem (Dress)

If M is a p-projective global Mackey functor, then tr is surjective, res is injective, and

$$M(G) \cong \operatorname{Im}(M([G])\colon M(S) \to M(S))$$

can be calculated by stable elements.

Stable elements:

Let $\mathcal{F} = \mathcal{F}_S(G)$ be the fusion system of G over S.
Category with
-Objects: Subgroups of S
-Morphisms: Conjugations in G.

Definition

The module (ring) of $\mathcal{F}$-stable elements in $M(S)$ is

$$M(\mathcal{F}) := \lim_{\mathcal{F}_S(G)} M$$
$$\cong \{x \in M(S) \mid \varphi^*(x) = res(x) \forall P \leq S, \varphi \in \mathrm{Hom}(P, S)\}$$

Depends only on the fusion system!
Extend to abstract fusion systems?

S finite p-group

Definition

A fusion system on S is a category with:
-Objects: Subgroups of S.
-Morphisms satisfy

$$\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q).$$

Definition

A fusion system is saturated if it satisfies two additional axioms, playing the role of Sylow theorems.
I "prime to p axiom"
II "Extension axiom"

Definition (Linckelmann-Webb)

A characteristic biset for $\mathcal{F}$ is an (S, S)-biset Ω such that

a) $|\Omega|/|S| \equiv 1 (\text{mod}\, p)$

b) For all $P \leq S$ and $\varphi \in \text{Hom}_{\mathcal{F}}(P, S)$,
$\Omega \circ [P, \varphi]_P^S = \Omega \circ [P, \text{incl}]_P^S$ (right $\mathcal{F}$-stable),
and
$[\varphi(P), \varphi^{-1}]_S^P \circ \Omega = [P, \text{id}]_S^P \circ \Omega$ (left $\mathcal{F}$-stable).

c) Ω lies in the span of $\{[P, \varphi] \mid P \leq S, \varphi \in \text{Hom}_{\mathcal{F}}(P, S).\}$

Motivation:

b) $xG = G = Gx$ for $x \in G$.

c) $[G]_S^S = \sum_{x \in S \backslash G / S} [S \cap S^x, c_x]_S^S$

Theorem (Broto–Levi–Oliver)

Every saturated fusion system has a characteristic biset.

Theorem (Linckelmann–Webb,Broto–Levi–Oliver)

Let H^* be cohomology with $\mathbb{F}_p$-coefficients. Then

$$H^*([\Omega])\colon H^*(S) \to H^*(S)$$

is an idempotent with image $H^*(\mathcal{F})$.

The theorem allows one to regard $H^*([\Omega])$ as a transfer map $H^*(S) \to H^*(\mathcal{F})$.

Diaz–Glesser–Mazza–Park: Replace $\mathbb{F}_p$ with abelian p-group.

What about other Mackey functors?
-$M([\Omega])$ generally not idempotent
-$[\Omega]$ is not unique

Definition

A characteristic idempotent for $\mathcal{F}$ is an idempotent ω in $\mathbb{Z}_{(p)} \otimes A(S, S)$ such that

a) $|\omega|/|S| = 1$
b) ω is right and left $\mathcal{F}$-stable
c) ω lies in the span of $\{[P, \varphi] \mid P \leq S, \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}$

Theorem (KR)

Every saturated fusion system has a unique characteristic idempotent.

Proof.

- If Ω characteristic biset, then Ω^n is also one.
- Some power of Ω is idempotent $\pmod p$ (since $\mathbb{F}_p \otimes A(S, S)$ is finite).
- Take characteristic biset Ω that is idempotent $\pmod p$. Then Ω^{p^n} is idempotent $\pmod p$.
- Conclude that $\Omega, \Omega^p, \Omega^{p^2}, \ldots$ is a Cauchy sequence converging to an idempotent ω in $\mathbb{Z}_p^\wedge \otimes A(S, S)$.
- Hard part: Coefficients in basis decomposition of ω satisfy fully determined system of equations, giving uniqueness.
- Since equations have integer coefficients, ω lies in $\mathbb{Z}_{(p)} \otimes A(S, S)$.

$\square$

The "hard part" involves describing $\omega_2 \mathbb{Z}_p^\wedge \otimes A(S_1, S_2)\omega_1$ for characteristic idempotents of fusion systems $\mathcal{F}_1$ and $\mathcal{F}_2$ over S_1 and S_2. Basically, multiplying by ω "quotients out" $\mathcal{F}$-conjugacy.

The description has other consequences.

Corollary (KR,Castellana–Morales)

For any p-local Mackey functor M, the map

$$M([\omega])\colon M(S) \to M(S)$$

is an idempotent with image $M(\mathcal{F})$.

Hence we can regard $M([\omega])$ as a transfer map $M(S) \to M(\mathcal{F})$.

This transfer map is unique, and natural with respect to fusion-preserving homomorphisms

Definition

For $X \in A(S, S)$, the stabilizer fusion system of X is the fusion system $\mathrm{Stab}(X)$ on S with morphism sets

$$\{\varphi \in (\, Hom)(P, Q) \mid X \circ [P, \varphi]_P^S = X \circ [P, \mathrm{incl}]_P^S\}$$

Corollary (KR,Puig?)

If Ω is a characteristic biset (or idempotent) for $\mathcal{F}$, then $\mathrm{Stab}(\Omega) = \mathcal{F}$.

This has an interesting interpretation in stable homotopy: We can recover $\mathcal{F}_S(G)$ from the stable homotopy type of the map $BS \to BG_p^\wedge$, but not from the stable homotopy type of $BG_p^\wedge$ [Martino–Priddy].

Saturation can also be detected in the Burnside ring.

Theorem (Puig,KR–Stancu)

Let $\mathcal{F}$ be a fusion system on S. If $\mathcal{F}$ has a characteristic biset (or idempotent), then $\mathcal{F}$ is saturated.

This is a (first?) radically different formulation of saturation.

The proof goes by counting fixed points.

For each $[P, \varphi]$, we have a fixed-point homomorphism

$$\chi_{[P,\varphi]}\colon A(S,S) \to \mathbb{Z},\ X \to |X^{\Delta_P^\varphi}|.$$

By tom Dieck, this gives an injection

$$A(S,S) \to \prod_{[P,\varphi]} \mathbb{Z}.$$

Condition b) becomes

$$\chi_{[P,\varphi]}(\Omega) = \chi_{[\varphi(P),\mathrm{incl}]}(\Omega)$$

$$\chi_{[P,\varphi]}(\Omega) = \chi_{[P,\mathrm{incl}]}(\Omega)$$

Condition c) becomes

$$\chi_{[P,\varphi]}(\Omega) = 0$$

when $\varphi \notin \mathcal{F}$.

For (S, S)-bisets X and Y, let $(X \times Y) \circ \Delta$ be the set $(X \times Y)$ regarded as an $(S, S \times S)$-biset via

$$(a_1, a_2)(x, y)b = (a_1 x b, a_2 y b)$$

for $(a_1, a_2) \in S \times S, (x, y) \in X \times Y, b \in S$.

Corollary (KR–Stancu)

If $\Omega \in A(S, S)$ satisfies the Frobenius reciprocity relation

$$(\Omega \times \Omega) \circ \Delta = (\Omega \times 1) \circ \Delta \circ \Omega,$$

then Ω is a characteristic biset for $\mathrm{Stab}(\Omega)$*.*

Proof.

The relation implies

$$\chi_{[Q,\psi]}(\Omega)\chi_{[Q,\varphi]}(\Omega) = \chi_{[\varphi(Q),\psi\circ\varphi^{-1}]}(\Omega)\chi_{[Q,\varphi]}(\Omega)$$

□

We should be able to prove corresponding statement for idempotents. Then we will have a bijection

$$\{\text{Frobenius idempotents in } \mathbb{Z}_{(p)} \otimes A(S,S)\}$$

$$\updownarrow$$

$$\{\text{Saturated fusion systems over } S\}$$