**Historical note:** Let $G$ be a finite group.

Atiyah (ca. 1960): There is an isomorphism

$$R(G)^{\wedge}_{I} \longrightarrow KU(BG),$$

where $R(G)$ is the complex representation ring and $I$ is the kernel of the augmentation

$$R(G) \to \mathbf{Z}, \quad V \mapsto \dim(V).$$

Segal conjectured that the analogous result holds for stable cohomotopy.

**Segal conjecture (weak form)**: There is an isomorphism

$$A(G)^{\wedge}_{I} \longrightarrow \pi^{0}_{S}(BG_{+}) := \{BG_{+}, S^{0}\},$$

where $A(G)$ is the Burnside ring of finite $G$-sets, and $I$ is the kernel of the augmentation

$$A(G) \to \mathbf{Z}, \quad X \mapsto |X|.$$

Lin: Proved conjecture for $G = \mathbf{Z}/2$.
Gunawardena: $G = \mathbf{Z}/p$, $p$ odd prime.
Ravenel: General finite cyclic groups.
Carlsson: Elementary abelian 2-groups.
Adams-Gunawardena-Miller:
Odd elementary abelian groups.
May-McClure: Reduce question to finite $p$-groups.
Carlsson: Uses A-G-M result and induction to prove $p$-group case.

Lewis-May-McClure: extended result to maps between classifying spaces. That is, replace $S^0$ by $BG'$.

**“Double Burnside modules”**

$\mathrm{Mor}(G, G')$ := Set of isomorphism classes of finite sets with right $G$-action and free left $G'$-action such that the actions commute.

$\mathrm{Mor}(G, G')$ is a monoid under disjoint union.

$A(G, G')$ := Grothendieck group completion of $\mathrm{Mor}(G, G')$.

$A(G, G')$ is a free module with one basis element $[H, \varphi]_G^{G'}$ for each conjugacy class of pairs $(H, \varphi)$, where $H \leq G$, $\varphi\colon H \to G'$.

Conjugacy is taken in both source and target. That is $(H_1, \varphi_1) \sim (H_2, \varphi_2)$ if and only if $\exists g \in G$, $\exists g' \in G'$ s.t.

$$\begin{array}{ccc} H_1 & \xrightarrow{\varphi_1} & \varphi_1(H_1) \\ \cong \big\downarrow c_g & & \cong \big\downarrow c_{g'} \\ H_2 & \xrightarrow{\varphi_2} & \varphi_2(H_2). \end{array}$$

The basis element $[H,\varphi]_G^{G'}$ corresponds to the biset

$$G \times G' / \Delta_H^{\varphi},$$

where

$$\Delta_H^{\varphi} = \{(h, \varphi(h) \mid h \in H\},$$

and actions are given by

$$b(x,y)a = (a^{-1}x, by),$$

for $a, x \in G$ and $b, y \in G'$.

The modules $A(G,G')$ form the morphism sets in a pre-additive category $\underline{A}$, whose objects are the finite groups, and where composition is given by

$$A(G',G'') \times A(G,G') \longrightarrow A(G,G''),$$

$$(\Omega',\Omega) \mapsto \Omega' \circ \Omega := \Omega' \times_{G'} \Omega.$$

This can be described on basis elements by the double coset formula:

$$[H',\varphi']_{G'}^{G''} \circ [H,\varphi]_{G}^{G'}$$

$$= \sum_{x \in H' \backslash G' / \varphi(H)} \left[\varphi^{-1}\left(\varphi\left(H\right) \cap H'^{x}\right), \varphi' \circ c_x \circ \varphi\right]_{G}^{G''}.$$

In particular, $A(G,G)$ is a ring, called the *double Burnside ring* of $G$.

There is also a pairing

$$A(G, 1) \times A(G, G') \longrightarrow A(G, G')$$

$$(X, \Omega) \mapsto X \cdot \Omega := X \times \Omega$$

where $G$ acts via the diagonal, and $G'$ acts on the second coordinate.

This makes $A(G) = A(G, 1)$ into a ring, called the *Burnside ring* of $G$, which acts on $A(G, G')$.

We have an "augmentation functor"

$$\epsilon : \underline{A} \longrightarrow \mathbb{Z}$$

defined on $(G, G')$-bisets by

$$\epsilon(\Omega) = |\Omega/G'|.$$

Functoriality means composition is sent to multiplication.

Note that

$$\epsilon([H,\varphi]_G^{G'}) = \left| \left( \frac{G \times G'}{\Delta_H^{\varphi}} \right) /G' \right| = |G/H|.$$

On $A(G)$, this gives the augmentation. We let $I(G)$ be the augmentation ideal

$$I(G) = \mathrm{ker}(\epsilon : A(G) \to \mathbb{Z}, X \mapsto |X|).$$

We are interested in the functor

$$\alpha : \underline{A} \longrightarrow \mathcal{S}pectra$$

acting on objects by

$$G \mapsto \mathbb{B}G_+,$$

where

$$\mathbb{B}G_+ := \Sigma^\infty(BG_+) \simeq \Sigma^\infty BG \vee \mathbb{S}^0,$$

and acting on morphisms by

$$[H, \varphi]_G^{G'} \mapsto \mathbb{B}\varphi_+ \circ tr_H,$$

$$\mathbb{B}G_+ \xrightarrow{tr_H} \mathbb{B}H_+ \xrightarrow{\mathbb{B}\varphi_+} \mathbb{B}G'_+.$$

Lewis-May-McClure showed that the following is a consequence of Carlsson's proof of the Segal conjecture.

**Theorem** (Segal conjecture). *$\alpha$ induces an isomorphism*

$$A(G, G')^{\wedge}_{I} \longrightarrow \{BG_+, BG'_+\},$$

*where $I = I(G)$ is the augmentation ideal of the Burnside ring $A(G)$.*

With the appropriate definition of $A(G, G')$, we can let $G'$ be a compact Lie group.

The $I(G)$-adic completion can be difficult to calculate. However, in the special case where $G$ is a $p$-group, May-McMclure noticed that the $I(G)$-adic topology coincides with the $p$-adic topology, at least after removing basepoints.

Recall that $\Sigma_+^\infty BG' \simeq \Sigma^\infty BG' \vee \mathbb{S}^0$. We have

$$\{BG, BG'\} \cong \{BG_+, BG'_+\}/\{BG_+, \mathbb{S}^0\}.$$

Put

$$\widetilde{A}(G, G') := A(G, G')/\langle [H, triv]_G^{G'} \mid H \leq G' \rangle.$$

Then we get an induced natural map

$$\alpha : \widetilde{A}(G, G') \longrightarrow \{BG, BG'\}.$$

**Theorem** (Lewis-May-McClure,Carlsson)**.** *Let $S$ be a finite $p$-group and $G'$ be a finite group. Then $\alpha$ induces an isomorphism*

$$\widetilde{A}(S, G')_p^\wedge \longrightarrow \{BS, BG'\},$$

*where $(-)_p^\wedge = \mathbb{Z}_p^\wedge \otimes (-)$ is $p$-adic completion.*

Again, $G'$ can be a compact Lie group.

Let $A_p(G, G')$ be the submodule of $A(G, G')$ generated by $(G, G')$-bisets whose $G$-isotropy groups are $p$-groups. We have

$$A_p(G, G') = \langle [P, \varphi]_G^{G'} \mid P \leq G \ p-\text{subgroup} \rangle.$$

**Theorem** (KR)**.** *For finite groups $G$ and $G'$, $\alpha$ induces an isomorphism*

$$\widetilde{A}_p(G, G')_p^\wedge \longrightarrow \{BG_p^\wedge, BG'\}.$$

Again, $G'$ can be a compact Lie group.

Since

$$\mathbb{B}G \simeq \bigvee_p \mathbb{B}G_p^\wedge,$$

we get the following consequence.

**Corollary** (KR)**.** *For finite groups $G$ and $G'$, $\alpha$ induces an isomorphism*

$$\bigoplus_p \widetilde{A}_p(G,G')_p^\wedge \to \bigoplus_p \{BG_p^\wedge, BG'\} \cong \{BG, BG'\}.$$

This is arguably a simpler description of $\{BG, BG'\}$ than the $I(G)$-adic completion.

The map in the statement is actually the $p$-completion of the composite

$$\widetilde{A}_p(G,G') \xrightarrow{\alpha} \{BG, BG'\} \xrightarrow{\iota_p^*} \{BG_p^\wedge, BG'\},$$

where

$$\iota_p \colon \mathbb{B}G_p^\wedge \hookrightarrow \mathbb{B}G$$

is the natural wedge-summand inclusion obtained from the natural splitting $\mathbb{B}G \simeq \bigvee_q \mathbb{B}G_q^\wedge$

Note that the target is $p$-complete since

$$\{BG_p^\wedge, BG'\} \cong \bigoplus_q \{BG_p^\wedge, BG'^\wedge_q\} \cong \{BG_p^\wedge, BG'^\wedge_p\},$$

Pick a Sylow $p$-subgroup $S$ of $G$.

Let $R$ be the operation restricting $(G, G')$-bisets to $(S, G')$-bisets. This is linear, and we have a homomorphism

$$R : A(G, G') \longrightarrow A(S, G'),$$

$$\Omega \mapsto \Omega \times_G [G]_S^G = \Omega \circ [S, \iota_S]_S^G,$$

where $[G]_S^G$ is $G$ regarded as a $(S, G)$-biset.

Applying $\alpha$ we get the homomorphism

$$R : \{BG_p^\wedge, BG'\} \longrightarrow \{BS, BG'\},$$

$$f \mapsto f \circ \mathbb{B}\iota_S.$$

Let $T$ be the operation inducing a $(G,G')$-biset from a $(S,G')$-biset. This gives a homomorphism

$$T : A(S,G') \longrightarrow A(G,G'),$$

$$\Omega \mapsto \Omega \times_S [G]^S_G = \Omega \circ [S,id]^S_G,$$

where $[G]^S_G$ is $G$ regarded as a $(G,S)$-biset.

Applying $\alpha$ we get a homomorphism

$$T : \{BS, BG'\} \longrightarrow \{BG^\wedge_p, BG'\},$$

$$f \mapsto f \circ tr_S.$$

Since

$$\mathbb{B}\iota_S \circ tr_S : \mathbb{B}G_p^\wedge \to \mathbb{B}G_p^\wedge$$

acts as multiplication by $|G/S|$ in $H^*(-;\mathbb{F}_p)$, it is a homotopy equivalence.

Therefore,

$$T \circ R : \{BG_p^\wedge, BG'\} \longrightarrow \{BG_p^\wedge, BG'\}$$

is an isomorphism.

We now have a (partial) commutative diagram:

$$\begin{array}{ccc}
R \circ T(\widetilde{A}(S, G')_p^\wedge) & \xrightarrow[\cong]{\alpha} & R \circ T(\{BS, BG'\}) \\
\uparrow R & & \uparrow R \cong \\
 & & T(\{BS, BG'\}) \\
 & & \uparrow = \\
\widetilde{A}_p(G, G')_p^\wedge & \xrightarrow{\alpha} & \{BG_p^\wedge, BG'\}.
\end{array}$$

If we can show that $R$ restricts to a map as indicated by the dashed arrow on the left side of the diagram, and that the restriction is an isomorphism, we can conclude that the bottom $\alpha$ in the diagram is an isomorphism, completing the proof.

We prove this "before taking quotients". That is, we show that $R$ restricts to an isomorphism

$$A_p(G, G')_p^\wedge \xrightarrow{\cong} R \circ T(A(S, G')_p^\wedge)$$

It is easy to check that the isomorphism "survives" to $\widetilde{A}$.

$A_p(G,G')^\wedge_p$ has basis

$$\{[P_i,\varphi_i]_G^{G'} \mid i \in J\}.$$

By Sylow's theorems (and since $P_i$ is only determined up to conjugacy) we may assume that $P_i \leq S$. We can then consider

$$C = \{[P_i,\varphi_i]_S^{G'} \mid i \in J\}$$

and

$$M_C = \langle [P_i,\varphi_i]_S^{G'} \mid i \in J\rangle \subset A_p(S,G')^\wedge_p.$$

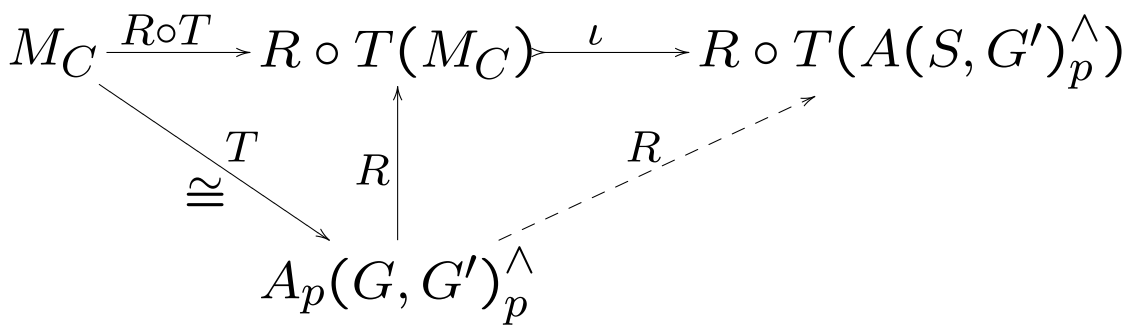
Note that $T$ maps $[P_i, \varphi_i]_S^{G'}$ to $[P_i, \varphi_i]_G^{G'}$. Therefore $T$ maps the basis $C$ of $M_C$ to a basis of $A_p(G, G')_p^{\wedge}$, and we have an isomorphism

$$T : M_C \longrightarrow A_p(G, G')_p^{\wedge}.$$

As illustrated on the diagram on the next page, we can conclude that $R$ induces a map

$$R : A_p(G, G')_p^{\wedge} \longrightarrow R \circ T(A_p(S, G')_p^{\wedge}).$$

Consider the diagram:




We are trying to show that the map $R$ represented by the dashed arrow (which we now see does exist) is an isomorphism. Looking at the diagram we see that it suffices to show that the maps $R \circ T$ and $\iota$ are isomorphisms.

**$R \circ T$ is an isomorphism:**

We say $[Q,\psi]_S^{G'} \precsim [P,\varphi]_S^{G'}$ if there exist $g \in G$ and $g' \in G'$ making the following diagram commute:

$$\begin{array}{ccc} Q & \xrightarrow{\psi} & \psi(Q) \\ \Big\downarrow c_g & & \Big\downarrow c_{g'} \\ P & \xrightarrow{\varphi} & \varphi(P). \end{array}$$

This is a transitive relation.

Say $[Q,\psi]_S^{G'} \sim [P,\varphi]_S^{G'}$ if

$$[Q,\psi]_S^{G'} \precsim [P,\varphi]_S^{G'} \text{ and } [P,\varphi]_S^{G'} \precsim [Q,\psi]_S^{G'}.$$

This is an equivalence relation, and $C$ is a set of representatives for equivalence classes.

Say $[Q,\psi]_S^{G'} \precnsim [P,\varphi]_S^{G'}$ if

$$[Q,\psi]_S^{G'} \precsim [P,\varphi]_S^{G'} \text{ but not } [P,\varphi]_S^{G'} \sim [Q,\psi]_S^{G'}.$$

The subconjugacy relation gives us two filtrations of $A_p(S,G')^\wedge_p$ as follows.

Put

$$M(\precsim [P,\varphi]) = \langle [Q,\psi] \precsim [P,\varphi]\rangle,$$

and

$$M(\precnsim [P,\varphi]) = \langle [Q,\psi] \precnsim [P,\varphi]\rangle.$$

**Lemma.** $R \circ T$ *preserves these filtrations. That is*

$$R \circ T(M(\precsim [P, \varphi])) \subset M(\precsim [P, \varphi])$$

*and*

$$R \circ T(M(\precnsim [P, \varphi])) \subset M(\precnsim [P, \varphi]).$$

*Proof.* Have

$$R \circ T([Q, \psi]_S^{G'}) = [Q, \psi]_S^{G'} \circ [S, id]_G^S \circ [S, \iota_S]_S^G.$$

Use double coset formula. $\square$

**Lemma.** $R \circ T$ *preserves stratification. That is*

$$R \circ T([P, \varphi]) \in M(\precsim [P, \varphi]) \setminus M(\precnsim [P, \varphi]).$$

*Proof.*

$$\epsilon(R \circ T([P, \varphi])) = \epsilon([P, \varphi]) \cdot \epsilon([G]_S^S) = \frac{|G|}{|S|} \cdot \epsilon([P, \varphi]),$$

but

$$\epsilon(M(\precnsim [P, \varphi])) = p \cdot \epsilon([P, \varphi])\mathbb{Z},$$

and $p \nmid \frac{|G|}{|S|}$. $\square$

We can now regard $R \circ T$ as an upper triangular matrix with nonzero entries on the diagonal on $M_C$. Hence we get an isomorphism

$$R \circ T : M_C \longrightarrow R \circ T(M_C).$$

$\iota$ **is iso:** Suffices to show surjectivity.

Every basis element $[P,\varphi]_S^{G'}$ of $A_p(S,G')_p^\wedge$ is $(G,G')$-conjugate to some $[P_i,\varphi_i]_S^{G'} \in C$.

Now

$T([P,\varphi]_S^{G'}) = [P,\varphi]_G^{G'} = [P_i,\varphi_i]_G^{G'} = T([P_i,\varphi_i]_S^{G'})$,

so

$$R \circ T([P,\varphi]_S^{G'}) = R \circ T([P_i,\varphi_i]_S^{G'}).$$

This completes the proof.