

Remarks on W. H. Young's “On Multiple Fourier Series”

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Abstract. W. H. Young presented simple sufficient conditions for the convergence and for the Cesàro summability of a double Fourier series at a fixed point. Observing that his results extend easily to d dimensions, he speculated that in general, once a result was properly extended to two variables, the complete generalization to any number of variables would follow. After describing his results, we argue that the situation in dimension $d \geq 3$ has turned out to be more complex than could have been guessed in the early parts of the twentieth century.

Introduction

This article is written to commemorate the appearance of the collected Fourier series writings of G. C. and W. H. Young.[CW]

When people first began to work on multiple trigonometric series, it was presumed that most results from the one dimensional theory would extend to higher dimensions by straightforward induction arguments. For example, as late as 1948, the great J. E. Littlewood gave voice to this opinion in a lecture at the University of Chicago. (Zhizhiashvili's survey article[Zh] written in the early 1970s has 609 references, which suggests that things did not turn out that way.) Young himself, although he was aware of some of the problems that might arise and treated them with care, subscribed to this thesis, at least to the extent of stating that once a result was properly extended to two variables, “the complete generalization for any number of variables” would follow. In the last section below, I will give a few examples to show that there is a much less uniform texture to this subject than anyone could have guessed. In any case, Young restricts himself to extensions where the n dimensional case is no more difficult than the two dimensional one. For this reason he presents all his results in the two dimensional setting. Thus his

general goal of extending the basic one dimensional Fourier series pointwise convergence results of Dirichlet-Jordan and Fejér to n dimensions is reached by extending them to two dimensions in a manner that clearly generalizes.

Young's results

Let us start within the context of classical one dimensional Fourier series. Here we are given a real-valued function $f \in L^1[-\pi, \pi]$ and its corresponding Fourier series

$$S[f] := \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt$ and $b_n := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt$.

There are two important conditions that deal with the question of whether one can recover the value of f back from $S[f]$ at a single point x . Let

$$S_n(x) := \frac{1}{2}a_0 + \sum_{\nu=1}^n (a_{\nu} \cos \nu x + b_{\nu} \sin \nu x)$$

be the n -th partial sum of the Fourier series and let

$$\sigma_n(x) := \frac{1}{n+1} \sum_{\nu=0}^n S_{\nu}(x)$$

be the Cesàro means of the Fourier series. Here are two theorems which answer the question.

Theorem 1 If f is continuous at x and of bounded variation on $[-\pi, \pi]$, then $\lim_{n \rightarrow \infty} S_n(x) = f(x)$.

Theorem 2 If f is continuous at x , then $\lim_{n \rightarrow \infty} \sigma_n(x) = f(x)$.

On the one hand, Theorem 1 (the Dirichlet-Jordan test) has a prettier conclusion; but on the other hand, Theorem 2 (Fejér's) has a prettier hypothesis. These theorems have been somewhat strengthened over the years (see, e.g., [Zy], Vol. I, p.57 and [Zy], Vol. I, pp.89, 94), but they give a very good idea of what to expect. Young moves to the context of two dimensional Fourier series and offers three generalizations of them. One is like the former, one is like the latter, and one falls in between.

To discuss what he has done, we need some definitions. Let $\mathbb{T}^2 := [-\pi, \pi] \times [-\pi, \pi]$. The Fourier series of a real-valued $F \in L(\mathbb{T}^2)$ is given by

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$$S[F] := \frac{1}{4}a_{00} + \frac{1}{2} \sum_{m=1}^{\infty} (a_{m0} \cos mx + d_{m0} \sin mx) + \frac{1}{2} \sum_{n=1}^{\infty} (a_{0n} \cos ny + c_{0n} \sin ny) +$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn} \cos mx \cos ny + b_{mn} \sin mx \sin ny + c_{mn} \cos mx \sin ny + d_{mn} \sin mx \cos ny),$$

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where $a_{mn} := \pi^{-2} \int_{\mathbb{T}^2} F(x, y) \cos mx \cos ny \, dx dy$, et cetera. For brevity, freeze $\mathbf{x} := (x, y)$ and write B_{mn} for the general term of this series. Following the custom of his time, Young set

$$S_{mn} := \sum_{\mu=0}^m \sum_{\nu=0}^n B_{\mu\nu}$$

of whether one can

and said that $S[F]$ was convergent at \mathbf{x} if $\lim_{\min\{m,n\} \rightarrow \infty} S_{mn}$ existed. Today we call such convergence unrestricted rectangular convergence. It is still of great interest, but other methods of convergence such as spherical, square, and restricted rectangular, have become so important that it has become a common practice to justify one's choice of convergence method and/or to make comparisons between methods. See [As] or [AW] for more on this.

Along with the rectangular partial sums, Young considered the Cesàro partial sums

$$\sigma_{mn} := \frac{1}{m+1} \frac{1}{n+1} \sum_{\mu=0}^m \sum_{\nu=0}^n S_{\mu\nu} \text{ and the mixed partial sums } s_{mn} := \frac{1}{m+1} \sum_{\mu=0}^m S_{\mu n}.$$

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We have three corresponding methods of summability:

then $\lim_{n \rightarrow \infty} S_n(x) =$

say that F is $(C, 1, 1)$ summable at \mathbf{x} if $\lim_{\min\{m,n\} \rightarrow \infty} \sigma_{mn}$ exists;

say that F is $(C, 1, 0)$ summable at \mathbf{x} if $\lim_{\min\{m,n\} \rightarrow \infty} s_{mn}$ exists;

say that F is $(C, 0, 0)$ summable at \mathbf{x} if $\lim_{\min\{m,n\} \rightarrow \infty} S_{mn}$ exists.

It is clear (from considering the trivial case $F(x, y) = f(x)$, for example), that just like the one dimensional case, one will need stronger hypotheses as one moves from the weak $(C, 1, 1)$ conclusion to the strong $(C, 0, 0)$ one. Here are Young's three theorems:

Theorem 3 If F is continuous at $\mathbf{x} \in \mathbb{T}^2$ and bounded on a cross-neighborhood of \mathbf{x} , then F is $(C, 1, 1)$ summable at \mathbf{x} .

Theorem 4 If F is continuous at $\mathbf{x} \in \mathbb{T}^2$ and bounded on a cross-neighborhood of \mathbf{x} , and if $F(\cdot, y)$ is continuous at x and if $F(x, \cdot)$ is of bounded variation, then F is $(C, 1, 0)$ summable at \mathbf{x} .

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$= [-\pi, \pi] \times [-\pi, \pi]$.

Theorem 5 If F is continuous at $x \in \mathbb{T}^2$ and of bounded variation on a cross-neighborhood of x , then F is $(C, 0, 0)$ summable at x .

To understand the statements of these theorems we need two further definitions. A cross neighborhood of x is a set of the form $(x - \delta, x + \delta) \times \mathbb{T} \cup \mathbb{T} \times (y - \epsilon, y + \epsilon)$. See page 90 of [As] for a picture of one. Young, who calls it a "crucial neighborhood," may very well have been the first person to see that such neighborhoods would play a crucial(!) role in multiple trigonometric series. For example, the principle of localization, which says that the behavior of a Fourier series at a point depends only on the behavior of the function itself near that point, extends easily to higher dimensions only if "near" is interpreted in terms of cross neighborhoods [As].

Young uses (and even improves) the definition of bounded variation for functions of two variables given earlier (and independently) by Krause and Hardy. Because of Young's work here, bounded variation can usefully be defined as follows. For a rectangle $R = [a, b] \times [c, d]$ let $F(R) := F(b, d) - F(a, d) - F(b, c) + F(a, c)$. If $\wp = \{R_i\}$ is a partition of R into subrectangles with sides parallel to the axes, let $V(\wp) := \sum |F(R_i)|$. Finally, say that F is of bounded variation on R if all three of (1) $\sup_{\wp} V(\wp)$, (2) the one dimensional variation of $F(\cdot, c)$ on $[a, b]$, and (3) the one dimensional variation of $F(a, \cdot)$ on $[c, d]$ are finite.

In order to carry out his proof, Young extends several of the standard one dimensional Fourier series tools to higher dimensions. Two of these that are quite important are the Riemann-Lebesgue theorem and Tauber's theorem. Young's two dimensional version of the Riemann-Lebesgue theorem states that the mn Fourier coefficients tend to zero as $\min\{|m|, |n|\} \rightarrow \infty$. Young presents several Tauberian conditions and shows, roughly speaking, that when any one of them is enjoyed by a double numerical series having any form of unrestricted rectangular Cesàro convergence, then the series is also unrestrictedly rectangularly convergent.

Some historical perspective

There has turned out to be an inhomogeneity of results, a sort of dependence upon dimension in the last 30 years that could not have been predicted. A major contributor to this was the decidedly one dimensional theorem of Carleson-Hunt which asserts that when $p > 1$ and $f \in L^p(\mathbb{T})$, $S[f]$ converges almost everywhere. This allowed C. Fefferman, P. Sjölin, and N. R. Tevzadze to independently prove that when $p > 1$ square convergence of Fourier series does occur almost everywhere in all dimensions; but C. Fefferman also showed that almost everywhere restricted rectangular convergence of the Fourier series may fail, even for a continuous function in dimension 2. (See [As] for references.) Also on the negative side, an easy consequence of a spectacular result of C. Fefferman[Fe] is that almost everywhere spherical convergence need not occur for $L^p(\mathbb{T}^n)$ functions if $n \geq 2$ and $p < 2$; but what happens here if $p \geq 2$ is still an open question.

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In dimension 2, for any $\epsilon > 0$, the circular Cesàro means of order ϵ converge in L^p norm for all p in the critical range $[\frac{4}{3}, 4]$, but whether the same thing happens at the endpoints $\frac{3}{2}$ and 3 of the corresponding critical range in dimension 3 is open. See [Co] for this and for additional results which don't extend from 2 to 3 dimensions. An important result of Roger Cooke[Co] is a sort of reverse Hölder inequality asserting that if $A_k(x, y) := \sum_{m^2+n^2=k} c_{mn} e^{i(mx+ny)}$, then $\|A_k\|_4 \leq \|A_k\|_2$. No three dimensional analogue is known and there may very well not be one. Spectral synthesis is another area where a good thing happens for the last time in dimension two. In particular, the circle is a set of spectral synthesis and the sphere is not[He].

For each mode of convergence, there are many questions about uniqueness of representation by a multiple trigonometric series. For example, the simplest question connected to the kind of convergence studied by Young is the question of whether a series which is unrestrictedly rectangularly convergent to zero at every point must have every coefficient equal to zero. Although this question has a positive answer in all dimensions, the first two dimensional proof was not extendable to higher dimensions. The two distinct general inductive proofs that were found many years later required quite different ideas.[AFR],[T],[AWa]

Turning to another mode of convergence, there is a two dimensional uniqueness theorem of Victor Shapiro and Roger Cooke stating that if a general trigonometric series is circularly convergent to 0, then all its coefficients must be 0. The corresponding question in 3 dimensions remained open for many years, finally yielding to an attack of Bourgain. But the $d \geq 3$ theorem required brownian motion, harmonic measure, and several other tools that were not necessary to achieve the Shapiro-Cooke result.[Bo1], [As1], [AWa]

Some very dimensionally dependent things happen in the arena of localization also. For example, the Fourier series of the characteristic function of the unit ball is spherically convergent at the origin (to the value 1) only in dimensions 1 and 2; if dimension exceeds 2, divergence occurs.[PST] Finally, if an $L^1(\mathbb{T}^2)$ function is 0 in a two dimensional interval I , then it is $(C, 1, 1)$ summable almost everywhere on I ; but there is an $L^1(\mathbb{T}^3)$ function which is 0 in a three dimensional interval N and not summable $(C, 1, 1, 1)$ at any point of N . (See [Zy], Vol. II, p.329, exercises 4 and 5.)

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