

A Characterization of Isometries

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THEOREM. *If a linear operator preserves the L^2 norm of the characteristic function of every interval on \mathbf{R} , then it is a real isometry on $L^2(\mathbf{R})$.* A counterexample shows that \mathbf{R} may not be replaced by \mathbf{R}^2 in the theorem. Other counterexamples show that if we replace "preserves" by "decreases" in the hypothesis of the theorem, then T may fail to be bounded.

1. INTRODUCTION

A major tool for studying bounded linear operators on $L^2(\mathbf{R}^n)$ is the Fourier transform. In case an operator does not commute with translations, however, other methods are often needed. The characterization we give here provides an interesting and practical alternative method which treats operators that do not necessarily commute with translations. We must pay for this gain in generality—our linear operator T must be an isometry.

DEFINITION. A linear map $T: \text{Re } L^2(\mathbf{R}^n, d\mu) \rightarrow \mathcal{H}$ (\mathcal{H} is a complex Hilbert space) is an *isometry* provided $\|Tf\| = \|f\|_2 = (\int_{\mathbf{R}^n} f^2(x) d\mu(x))^{1/2}$ for all f in $\text{Re } L^2(\mathbf{R}^n, d\mu)$. Thus, T is a distance-preserving map in the sense of point set topology.

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The main result of this paper, in essence, is the following characterization:

If a linear operator preserves the L^2 norm of the characteristic function of every interval of the real line \mathbf{R} , then it is an isometry on $L^2(\mathbf{R})$. (For a more precise statement see Theorem 1.)

Many plausible generalizations of this result fail to be true. The first surprise is that we cannot replace the line by the plane (where interval means rectangle with sides parallel to the axes). For this, see Section 3, Counterexample 1 which depends on this simple but little known geometric fact:

If a rectangle is divided into four disjoint subrectangles by two perpendicular lines parallel to its sides, then the product of the areas of one pair of opposite subrectangles is equal to the product of the other two areas.

If we attempt to extend the characterization from isometries to bounded operators, the following generalization springs to mind: If a linear operator shrinks the L^2 norm of the characteristic function of every interval of the real line \mathbf{R} , then it is a contraction on $L^2(\mathbf{R})$; but this fails to be true. Counterexample 2 of Section 3 is an unbounded linear operator satisfying the hypothesis of this proposed generalization. There is already known an unbounded linear operator which shrinks the norm of the characteristic function of every measurable set and thus is of restricted type (2, 2). Stein and Weiss introduced this example in [6, pp. 283–284] to distinguish bounded maps from L^p to L^q from maps of restricted type (p, q) . Counterexample 2 is of independent interest because it is a convolution operator.

Theorem 3, although only a special case of Theorem 1, has an entirely different proof which is of independent interest. (It preceded the more functional analytic Theorem 1.) One tool it uses is this interesting formula for C_0^1 functions:

$$\int_{-\infty}^{\infty} f^2(t) dt = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |t-s| f'(s) f'(t) ds dt \quad (\text{Theorem 2, Section 2}).$$

Not every extension of the characterization fails; if the class of intervals is extended to include certain extra sets, a useful form of the characterization for $\text{Re } L^2(\mathbf{R}^2)$ is obtained (Theorem 4, Section 3). This can be used to give very fast proofs of some well-known facts about tensor products of isometries (see [3]).

Finally, the characterization is applied to produce new proofs that the Fourier and Hilbert transforms are unitary on $L^2(\mathbf{R}^n)$. (A related proof that the Fourier transform is unitary uses Bochner's characterization of operators unitary on $L^2(\mathbf{R}, dx)$ [5, pp. 291–295].) Thus, one can show that the Hilbert transform is unitary without using Fourier transform techniques.

Another application, which we give elsewhere [1, 2], is that convolution with

$$k_\gamma(x) = \left[\frac{2\pi}{\gamma} \tanh\left(\frac{\pi\gamma}{2}\right) \right]^{-1/2} \frac{\text{sgn } x}{|x|^{1+i\gamma}}$$

is a unitary operator. The kernels k_γ were studied by Muckenhoupt [4]. As $\gamma \rightarrow 0$, $k_\gamma * f$ converges to the Hilbert transform in L^p , $1 < p < \infty$, but not pointwise a.e. [1]. This is connected with the unboundedness of the associated maximal operator, a fact closely related to Counterexample 2.

2. BASIC RESULTS

We caution the reader that an isometry $T: \text{Re } L^2(\mathbf{R}^n, d\mu) \rightarrow \mathcal{H}$ in our sense may not remain an isometry when T is extended in the obvious manner to complex-valued functions. In fact, it may not even be true that $(Tf, Tg) = (f, g) = \int_{\mathbf{R}^n} f(x) \overline{g(x)} d\mu(x)$ for real-valued f and g , where the inner product on the left-hand side is that of \mathcal{H} . On the other hand, $(f, g) = \text{Re}(Tf, Tg)$ follows immediately from the polarization identity.

THEOREM 1. *A linear transformation that preserves L^2 norm for the characteristic function of each interval is an isometry on $L^2(\mathbf{R})$.*

More precisely, suppose μ is a Borel measure on the reals which is finite on bounded intervals. Let SF be the space of real-valued step functions based on intervals. If T is a linear map defined on SF with values in some complex Hilbert space \mathcal{H} which is norm-preserving on characteristic functions of intervals (i.e., $\|T\chi_I\|^2 = \int_{-\infty}^{\infty} |\chi_I(t)|^2 d\mu(t) = \mu(I)$ for all intervals I), then T extends uniquely to an isometry from $\text{Re } L^2(\mathbf{R}, d\mu)$ to \mathcal{H} .

Proof. We present the proof when $d\mu$ is Lebesgue measure dx , since this case has more intuitive appeal. There are only two trivial changes in the proof of the general case: First, an interval must be defined as a bounded nonempty convex subset of \mathbf{R} (e.g., a point is then an interval), and second, additivity of measure must replace arguments such as $(c-b) + (b-a) = c-a$ (cf. (2.3)).

Linear combinations of the characteristic functions of intervals are dense in $\text{Re } L^2(\mathbf{R})$, so it suffices to prove that T preserves norm for such functions. If $f(x) = \sum_{i=1}^{\infty} a_i \chi_i(x)$, where the χ_i are the characteristic functions of intervals with disjoint interiors, then

$$\|f\|^2 = \sum a_i^2 \|\chi_i\|^2$$

and

$$\begin{aligned}\|Tf\|^2 &= \left(\sum_{i=1}^{\infty} a_i T\chi_i, \sum_{j=1}^{\infty} a_j T\chi_j \right) = \sum_{i=1}^{\infty} a_i^2 (T\chi_i, T\chi_i) + \sum_{i \neq j} a_i a_j (T\chi_i, T\chi_j) \\ &= \sum_{i=1}^{\infty} a_i^2 \|T\chi_i\|^2 + 2 \sum_{i < j} a_i a_j \operatorname{Re}(T\chi_i, T\chi_j).\end{aligned}\quad (2.1)$$

By hypothesis, $\|\chi_i\| = \|T\chi_i\|$, so it will suffice to show

$$\operatorname{Re}(T\chi_I, T\chi_J) = 0 \quad \text{for nonoverlapping intervals } I \text{ and } J. \quad (2.2)$$

First, consider the adjacent case: $\chi_I = \chi_{[a,b]}$, $\chi_J = \chi_{[b,c]}$. Then we have

$$(T\chi_I, T\chi_I) = (\chi_I, \chi_I) = \|\chi_I\|^2 = b - a, \quad (T\chi_J, T\chi_J) = c - b$$

and

$$\begin{aligned}(T(\chi_I + \chi_J), T(\chi_I + \chi_J)) &= (T\chi_I + T\chi_J, T\chi_I + T\chi_J) \\ &= (T\chi_I, T\chi_I) + (T\chi_J, T\chi_J) + 2 \operatorname{Re}(T\chi_I, T\chi_J) \\ &= (c - b) + (b - a) + 2 \operatorname{Re}(T\chi_I, T\chi_J) \\ &= (c - a) + 2 \operatorname{Re}(T\chi_I, T\chi_J).\end{aligned}\quad (2.3)$$

But $\chi_I + \chi_J$ is the characteristic function of the interval $[a, c]$, so

$$\|T(\chi_I + \chi_J)\|^2 = \|\chi_I + \chi_J\|^2 = c - a.$$

Combining this with (2.3), we see $\operatorname{Re}(T\chi_I, T\chi_J) = 0$.

Next, suppose the supports of χ_I and χ_J do not abut. Without loss of generality, suppose $\chi_I = \chi_{[a,b]}$, $\chi_J = \chi_{[c,d]}$, where $a < b < c < d$. The trick is to insert the intermediate characteristic function, $\chi_{[b,c]}$. We have

$$\|T\chi_{[a,d]}\|^2 = \|\chi_{[a,d]}\|^2 = d - a,$$

but

$$\begin{aligned}\|T\chi_{[a,d]}\|^2 &= \|T\chi_{[a,b]} + T\chi_{[b,c]} + T\chi_{[c,d]}\|^2 \\ &= (T\chi_{[a,b]}, T\chi_{[a,b]}) + (T\chi_{[b,c]}, T\chi_{[b,c]}) + (T\chi_{[c,d]}, T\chi_{[c,d]}) \\ &\quad + 2 \operatorname{Re}(T\chi_{[a,b]}, T\chi_{[b,c]}) + 2 \operatorname{Re}(T\chi_{[a,b]}, T\chi_{[c,d]}) \\ &\quad + 2 \operatorname{Re}(T\chi_{[b,c]}, T\chi_{[c,d]}) \\ &= (b - a) + (c - b) + (d - c) + 0 + 2 \operatorname{Re}(T\chi_I, T\chi_J) + 0,\end{aligned}$$

by three applications of the hypothesis and two applications of the preceding case. Thus, we have

$$d - a = d - a + 2 \operatorname{Re}(T\chi_I, T\chi_J), \quad \text{so } \operatorname{Re}(T\chi_I, T\chi_J) = 0,$$

completing the proof of (2.2). ■

Remarks. Some special cases of this theorem occur when the space on which the operators act is: (1) $\operatorname{Re} L^2(\mathbf{R})$ —this is what we proved above ($d\mu = dx$); (2) $\operatorname{Re} L^2$ of the circle ($d\mu = dx/2\pi$ on $[0, 2\pi)$); (3) the sequence space $\operatorname{Re} l^2$ ($d\mu =$ unit mass at each integer); and (4) finite (d , say)-dimensional Euclidean space ($d\mu =$ unit mass at the points $1, 2, \dots, d$).

Theorem 1 works because the “test” family of characteristic functions of intervals (we can reduce the family to the countable subfamily of intervals with rational endpoints) is sufficiently rich to determine the measure structure. No complete orthonormal family is so rich: In fact, if $\{\phi_j\}$ is such a family, set $T\phi_j = \phi_1$ for all j . Then $\|T\phi_j\| = \|\phi_1\| = 1 = \|\phi_j\|$ for all j , but T is certainly not an isometry.

We now proceed to the case of an integral transform. We start with a needed preliminary.

THEOREM 2. Suppose $f: \mathbf{R} \rightarrow \mathbf{R}, f \in C_0^1$ —the compactly supported continuously differentiable real-valued functions. Then

$$\|f\|^2 = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |t - s| f'(s) f'(t) ds dt. \quad (2.4)$$

Proof. Since the iterated integral may be considered as a double integral over \mathbf{R}^2 by Fubini's theorem and since the integrand is then symmetric in s and t ,

$$-\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |t - s| f'(s) f'(t) ds dt = \int_{-\infty}^{\infty} \int_{-\infty}^t (s - t) f'(s) f'(t) ds dt.$$

Integrating the inner integral by parts,

$$\int_{-\infty}^t (s - t) f'(s) ds = (s - t) f(s) \Big|_{-\infty}^t - \int_{-\infty}^t f(s) ds = - \int_{-\infty}^t f(s) ds = -F(t).$$

Thus, the iterated integral is equal to

$$- \int_{-\infty}^{\infty} f'(t) F(t) dt.$$

If we integrate this by parts, we get

$$-f(t)F(t) \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f(t)F'(t) dt = \int_{-\infty}^{\infty} f(t)^2 dt = \|f\|^2. \quad \blacksquare$$

Theorem 2 admits a generalization containing both Theorem 2 and a similar result for step functions as special cases. See [2] or [3] for details.

Although the following theorem is a special case of Theorem 1, the proof uses Theorem 2 and may be of independent interest.

THEOREM 3. *Let T be an integral transform defined by*

$$Tf(x) = \text{p.v.} \int_{-\infty}^{\infty} k(x, t) f(t) dt = \lim_{\epsilon \rightarrow 0} \int_{|x-t| > \epsilon} k(x, t) f(t) dt$$

for $f \in C_0^1(\mathbf{R})$, where k is a complex-valued function, jointly measurable in x and t . If k satisfies the condition

$$\int_{-\infty}^{\infty} \left| \lim_{\epsilon \rightarrow 0} \int_s^t k(x, u) du \right|^2 dx = |t - s| \quad \text{for all } s, t, \quad (2.5)$$

then T extends uniquely to an isometry from $\text{Re } L^2(\mathbf{R}, dx)$ to $L^2(\mathbf{R}, dx)$. Note that (2.5) means essentially that T preserves the L^2 norm of the characteristic function of each interval.

Proof. Since C_0^1 is a dense subset of $\text{Re } L^2$, it suffices to prove $\|Tf\| = \|f\|$ for $f \in C_0^1$. Define $K(x, t) = \text{p.v.} \int_0^t k(x, u) du$, and observe that (2.5) may be rewritten

$$\int_{-\infty}^{\infty} |K(x, t) - K(x, s)|^2 dx = |t - s| \quad \text{for all } s, t. \quad (2.6)$$

Fix x and write $K(x, t) = K(t)$. If we integrate the defining equation for $Tf(x)$ by parts and observe that f has compact support, we get

$$Tf(x) = - \int_{-\infty}^{\infty} K(t) f'(t) dt.$$

Taking complex conjugates,

$$\overline{Tf(x)} = - \int_{-\infty}^{\infty} \overline{K(s)} f'(s) ds.$$

Multiplying these last two equations we get

$$|Tf(x)|^2 = \int_{-\infty}^{\infty} K(t) f'(t) dt \int_{-\infty}^{\infty} \overline{K(s)} f'(s) ds.$$

Putting $s = 0$ into (2.6) shows that K is locally square integrable, thus locally integrable, so we may change the order of integration to get

$$|Tf(x)|^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K(t) \overline{K(s)} f'(t) f'(s) ds dt. \quad (2.7)$$

Since $|Tf(x)|^2$ is real, taking the complex conjugate of (2.7) does not change the value, and if we take the sum of the resulting equal expressions, we get

$$|Tf(x)|^2 = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [K(t) \overline{K(s)} + \overline{K(t)} K(s)] f'(t) f'(s) ds dt. \quad (2.8)$$

Now observe, if g and h are locally integrable functions of a single variable,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(t) f'(t) f'(s) ds dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(s) f'(t) f'(s) ds dt = 0,$$

since, for example, the first integral may be written as $\int_{-\infty}^{\infty} g(t) f'(t) (\int_{-\infty}^{\infty} f'(s) ds) dt$, and the inner integral is 0.

Thus, we may add $-|K(t)|^2 - |K(s)|^2$ to the expression in square brackets in (2.8) without changing the value of the integral to get

$$\begin{aligned} |Tf(x)|^2 &= -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [|K(t)|^2 - K(t) \overline{K(s)} - \overline{K(t)} K(s) + |K(s)|^2] \\ &\quad \times f'(t) f'(s) ds dt \\ &= -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(t) - K(s)|^2 f'(t) f'(s) ds dt. \end{aligned}$$

Recalling that $K(t) = K(x, t)$, integrate in x and change the order of integration to get

$$\int_{-\infty}^{\infty} |Tf(x)|^2 dx = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |K(x, t) - K(x, s)|^2 dx \right) f'(t) f'(s) ds dt.$$

First apply (2.6) and then Theorem 2 to get

$$\|Tf\|^2 = -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |t - s| f'(t) f'(s) ds dt = \|f\|^2. \quad \blacksquare$$

3. COUNTEREXAMPLES AND EXTENSIONS

It is a surprising fact that Theorem 1 does not generalize to \mathbf{R}^2 .

COUNTEREXAMPLE 1. There is a linear transformation that preserves the L^2 norm of the characteristic function of each interval in \mathbf{R}^2 , but is not an isometry on $L^2(\mathbf{R}^2)$. (Recall that an interval in \mathbf{R}^2 is the Cartesian product of two one-dimensional intervals.)

Proof. Let $S = I_1 \cup I_2 \cup I_3 \cup I_4$, where $I_1 = [0, 1] \times [0, 1]$, $I_2 = [-1, 0] \times [0, 1]$, $I_3 = [-1, 0] \times [-1, 0]$ and $I_4 = [0, 1] \times [-1, 0]$. Note that I_j is a square in the j th quadrant (see Fig. 1).

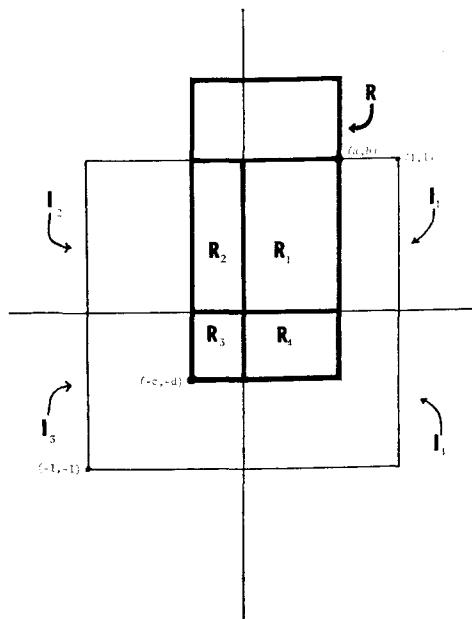


FIGURE 1

Let ϕ_j be the characteristic function of I_j . Since the four functions ϕ_j are orthonormal, they may be extended to a (real-valued) complete orthonormal basis of $L^2(\mathbf{R})$, $\{\phi_j | j = 1, 2, \dots\}$.

Define a linear operator T on $L^2(\mathbf{R})$ by $T\phi_3 = \phi_1$, $T\phi_4 = -\phi_2$, and $T\phi_i = \phi_i$ otherwise.

Now T is not an isometry, since $\|T(\phi_2 + \phi_4)\| = \|\phi_2 - \phi_2\| = 0$, while $\|\phi_2 + \phi_4\| = 2^{1/2}$. Next we show that T does preserve the norm of the charac-

teristic function χ of any interval R in \mathbf{R}^2 . Since $\chi \in L^2(\mathbf{R}^2)$, we may write $\chi = \sum a_i \phi_i$, $\|\chi\|^2 = \sum a_i^2$, so that

$$T\chi = \sum a_i T\phi_i = (a_1 + a_3)\phi_1 + (a_2 - a_4)\phi_2 + \sum_{i=5}^{\infty} a_i \phi_i,$$

$\|T\chi\|^2 = \sum_{i=1}^{\infty} a_i^2 + 2(a_1 a_3 - a_2 a_4)$. It only remains to show that

$$a_1 a_3 = a_2 a_4. \quad (3.1)$$

The rectangle $R \cap S = R_1 \cup R_2 \cup R_3 \cup R_4$, where the subrectangle $R_i = R \cap I_i$ is that portion of $R \cap S$ lying in the i th quadrant. Since for $i = 1, 2, 3, 4$,

$$a_i = (\chi, \phi_i) = \int_{R \cap I_i} dx = |R_i| = \text{area of } R_i;$$

(3.1) reduces to the geometric identity

$$|R_1| |R_3| = |R_2| |R_4| \quad (\text{see Fig. 1}). \quad (3.2)$$

If $R \cap S$ misses at least one quadrant, then it is easy to see that both sides of (3.2) are 0. Thus, we may let (a, b) be the vertex of R_1 opposite $(0, 0)$ and $(-c, -d)$ be the vertex of R_3 opposite $(0, 0)$ where a, b, c , and d are all positive. Thus,

$$|R_1| |R_3| = (ab)(cd) = (bc)(ad) = |R_2| |R_4|. \quad \blacksquare$$

The hypothesis that the norm be preserved is crucial to Theorems 1 and 3 in the following sense:

COUNTEREXAMPLE 2. There is a convolution operator K defined on SF which shrinks the norm of the characteristic function of each interval, but which is unbounded on $\text{Re } L^2(\mathbf{R})$.

Proof. We relent on our Fourier transform-less program for the duration of this example. Define K on SF by $Kf = k * f$, where

$$k(\xi) = \left(\frac{3}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} n \chi_{[n, n+n^{-1}]}(\xi),$$

so that $Kf(x) = F^{-1}[(Ff)(\xi) \cdot \hat{k}(\xi)](x)$, where F denotes the Fourier transform,

$$(Ff)(\xi) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx,$$

and F^{-1} denotes the inverse Fourier transform. Then since \hat{k} is unbounded,

K is an unbounded operator on L^2 [7, p. 28]. However, if χ is the characteristic function of an interval of length 2α , direct calculations yield

$$|(F\chi)(\xi)| = \left| \frac{1}{(2\pi)^{1/2}} \int_{c-\alpha}^{c+\alpha} e^{-ix\xi} dx \right| = \left(\frac{2}{\pi} \right)^{1/2} \left| \frac{\sin \alpha\xi}{\xi} \right|,$$

and $\|\chi\|^2 = 2\alpha$. Thus, by Plancherel's theorem

$$\begin{aligned} \|K\chi\|^2 &= \|F(K\chi)\|^2 = \int_{-\infty}^{\infty} |F\chi(\xi)|^2 k(\xi)^2 d\xi \\ &= \frac{6}{\pi^2} \int_{-\infty}^{\infty} \left(\frac{\sin \alpha\xi}{\xi} \right)^2 \sum n^2 \chi_{[n, n+n-4]}(\xi) d\xi \\ &= \frac{3}{\pi^2} \sum n^2 \int_{\alpha n}^{\alpha(n+n-4)} \left(\frac{\sin t}{t} \right)^2 dt \cdot 2\alpha \\ &\leq \frac{3}{\pi^2} \left(\sum \frac{1}{n^2} \right) \cdot 2\alpha = \frac{1}{2} \|\chi\|^2, \end{aligned}$$

where the integral is dominated by $\int_{\alpha n}^{\alpha(n+n-4)} dt$ if $\alpha \leq 1$ and by $\int_{\alpha n}^{\alpha(n+n-4)} (dt/t^2)$ if $\alpha > 1$. ■

Even in the finite-dimensional setting, where all linear transformations must be bounded, things go as wrong as they possibly can. For example, if P is a projection onto a subspace which does not contain any of the test vectors $(1, 0)$, $(0, 1)$, and $(1, 1)$, then $\|(1 + \epsilon)P\| > 1$ but $(1 + \epsilon)P$ shrinks the three test vectors if $\epsilon > 0$ is sufficiently small.

In view of Counterexample 1, it is not immediately clear how Theorem 1 should be generalized to higher dimensions. One possibility is to extend the test family of all intervals to a larger collection \mathcal{J} .

Let \mathcal{J} be the collection of all sets of the form $I \cup J$ where I and J are intervals of \mathbf{R}^n with a common "face" of dimension k , $0 \leq k \leq n-1$. For $n = 3$, the three basic shapes that elements of \mathcal{J} may have are shown in Fig. 2.

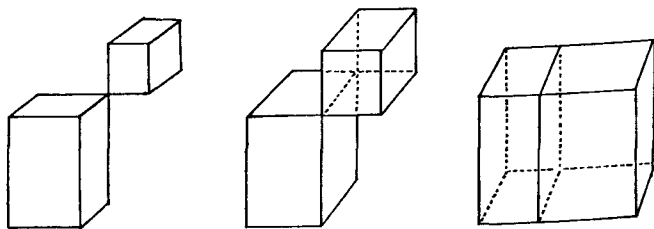


FIGURE 2

THEOREM 4. *If T is a linear operator defined on SF preserving the L^2 norm of the characteristic function of every element of \mathcal{J} , then T extends uniquely to an isometry of $\text{Re } L^2(\mathbf{R}^n, dx)$.*

Proof. We outline the proof for $n = 2$ since it displays the essence of the n -dimensional case. If we can prove (2.2) for I and J , then the result will follow from applying T to a linear combination of characteristic functions of intervals just as in the proof of Theorem 1. Now, given disjoint intervals I and J , extend their sides to form a (possibly degenerate) configuration of nine blocks R_{ij} , $1 \leq i, j \leq 3$, where R_{ij} is the block in the i th row and j th column. (For example, if I lies "northeast" of J one might have $I = R_{13}$, $J = R_{31}$, while if I lies due east of J and is smaller than J , one might have $I = R_{23}$, $J = R_{11} \cup R_{21} \cup R_{31}$.) To show (2.2) it suffices to prove $\text{Re}(T\chi_{R_{i,j}}, T\chi_{R_{i',j'}}) = 0$ if $(i, j) \neq (i', j')$, since χ_I and χ_J are each sums of the $\chi_{R_{i,j}}$ and the inner product is bilinear.

Any pair R, S of these nine intervals R_{ij} stands in one of the following relations:

- (i) They lie in the same row (or column) and touch.
- (ii) They lie in the same row (or column) and do not touch.
- (iii) They lie on the same diagonal and touch.
- (iv) They lie in adjacent rows (or columns) but do not touch. (In chess, a knight could move from one to the other.)
- (v) They lie on the same diagonal and do not touch.

We give the flavor of the remainder of the proof by proving case (v) under the assumption that the first four cases have been proved.

Say $R = R_{11}$ and $S = R_{33}$. Then $(R_{11} \cup R_{12} \cup R_{21} \cup R_{22}) \cup R_{33} \in \mathcal{J}$ and of the 10 terms of the form $2 \text{Re}(T\chi_{R_{i,j}}, T\chi_{R_{i',j'}})$ that appear in the expansion of $\|T\chi_{R_{11}} + T\chi_{R_{12}} + T\chi_{R_{21}} + T\chi_{R_{22}} + T\chi_{R_{33}}\|^2$, nine are immediately zero by the previous four cases, and hence, the tenth one, $2 \text{Re}(T\chi_{R_{11}}, T\chi_{R_{33}})$, must also be zero. ■

For some applications of Theorem 4, see [3].

When can Theorem 1 be used to characterize complex linear isometries of complex $L^2(\mathbf{R}, d\mu)$? Recall the cautionary remarks at the start of Section 2. Suppose T is an isometry from $\text{Re } L^2$ and T is extended to complex L^2 by $T(f + ig) = Tf + iTg$. Then

$$\begin{aligned} \|f + ig\|^2 &= \|f\|^2 + \|g\|^2 = \|Tf\|^2 + \|Tg\|^2 \\ &= \|T(f + ig)\|^2 - 2 \text{Im}(Tf, Tg), \end{aligned}$$

so to show T is an isometry from L^2 to a Hilbert space \mathcal{H} , it suffices to show that T is an isometry of $\text{Re } L^2$ as defined in the introduction, and that

$$\text{Im}(Tf, Tg) = 0 \quad \text{if } f \text{ and } g \text{ are real-valued.} \quad (3.3)$$

A simple sufficient condition for (3.3) to hold is the crux of the following lemma.

LEMMA. Suppose there exists an antiunitary operator C on \mathcal{H} such that $T(f) = CTf$ for all $f \in L^2(\mathbf{R}, d\mu)$. Then if T satisfies the hypotheses of Theorem 1, T is a complex isometry. (Recall that C antiunitary means that $C(\alpha f) = \bar{\alpha} Cf$ for $\alpha \in \mathbf{C}$ and $f \in \mathcal{H}$ and $(Cf, Cg) = (g, f)$. Complex conjugation is antiunitary on $L^2(\mathbf{R})$, for example.)

Proof. For real-valued f and g ,

$$\begin{aligned}(Tf, Tg) &= (T\bar{f}, T\bar{g}) = (CTf, CTg) \\ &= (Tg, Tf) \\ &= \overline{(Tf, Tg)},\end{aligned}$$

so

$$\operatorname{Im}(Tf, Tg) = 0. \quad \blacksquare$$

One can now give new proofs that the Fourier transform and the Hilbert transform are unitary operators. By means of the lemma and Theorem 3 (or Theorem 1) one shows that both transforms are isometries on $L^2(\mathbf{R})$. The adjoint F^* of the Fourier transform F is obviously given by $F^*f(x) = Ff(-x)$. Also, it may be shown that the adjoint H^* of the Hilbert transform H satisfies $H^* = -H$. The last two formulas show that F^* and H^* are likewise isometries, so that F and H are unitary on $L^2(\mathbf{R})$. This result extends easily to $L^2(\mathbf{R}^n)$ since each transform is simply the tensor product of n copies of the one-dimensional transform. See [3] for details.

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