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CONVERGENCE OF SERIES CONJUGATE TO A CONVERGENT MULTIPLE TRIGONOMETRIC SERIES

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RÉSUMÉ. — Si une série de Fourier double converge sans restriction rectangulaire dans un ensemble, alors sur cet ensemble la transformation de Riesz double, qui est formée par multiplication du (m, n)-ième terme par $mn/(m^2+n^2)$, converge de la même façon p. p. Si la fonction somme partielle maximale rectangulaire est finie sur un ensemble, alors sur cet ensemble la fonction g_λ^* associée est finie elle aussi p. p., pourvu que $\lambda \geqslant 28$.

ABSTRACT. — If a double trigonometric series converges unrestrictedly rectangularly on a set, then a. e. on that set the double Riesz transform, which is formed by multiplying the (m, n)-th term by $(mn)/(m^2+n^2)$, converges in the same way. If the maximal rectangular partial sun function is finite on a set, then a. e. on that set the associated g_λ^* function is also finite, provided $\lambda \geqslant 28$.

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1. Introduction

There is a one dimensional theorem of Plessner which states that the convergence of a trigonometric series on a set forces its conjugate series to converge almost everywhere on that set. In two dimensions the situation becomes much more complicated. First of all the notion of "converge" diverges into a number of quite distinct notions ([3], [1]). We will put aside this obstacle by (arbitrarily) choosing the mode of convergence to be unrestricted rectangular convergence (denoted herein as UR convergence). The second problem is that the notion of conjugate also diverges into several notions. In one dimension the conjugate of $\Sigma c_n e^{inx}$ is $\Sigma M_n c_n e^{inx}$ where the multiplier $M_n = -i \operatorname{sgn} n$. Given a two dimensional series $\Sigma c_{mn} e^{i(mx+ny)}$, the "conjugate" will again be $\Sigma M_{mn} c_{mn} e^{i(mx+ny)}$. Perhaps the most natural choice for M_{mn} would be $M_{mn}^{(1)} = -i \operatorname{sgn} m$ or $M_{mn}^{(2)} = (-i \operatorname{sgn} m)(-i \operatorname{sgn} n) = -\operatorname{sgn} mn$. However in [2] is given an a. e. UR convergent series whose conjugates in sense (1) or sense (2) both diverge a. e., even in the very relaxed sense of square convergence.

Other choices for the conjugate, introduced in [11], are

$$M_{mn}^{(3)} = \frac{m}{\sqrt{m^2+n^2}} \quad \text{and} \quad M_{mn}^{(4)} = \frac{mn}{m^2+n^2}.$$

These definitions are natural analogues of the Riesz transforms. More explicity, the continuous analogue of the multiplier $-i \operatorname{sgn} n$ is the Hilbert transform which may be expressed using the multiplier $-i \operatorname{sgn} x$. The 2 dimensional Riesz transforms then correspond to the multipliers

$$\frac{x_1}{\sqrt{x_1^2+x_2^2}} \quad \text{and} \quad \frac{x_2}{\sqrt{x_1^2+x_2^2}},$$

while the double Riesz transform corresponds to the multiplier $x_1 x_2/(x_1^2+x_2^2)$. The only trouble is that these multipliers are usually considered in the setting of circular convergence so that studying them from a rectangular point of view seems a bit strange. Nevertheless, Peterson and Welland were able to show that if a double series converges UR on a set, then the conjugates in both senses (3) and (4) converge *restrictedly* rectangularly a. e. on that set [11].

We improve their result for the multiplier $\{M_{mn}^{(4)}\}$, but can only underscore the difficulties associated with obtaining a similar improvement for the multiplier $\{M_{mn}^{(3)}\}$. Theorem 1 assumes the UR convergence of a double series on a set and concludes the a. e. UR convergence of the conjugate in sense (4) on that set. One of the most delicate parts of the proof of theorem 1 is a Tauberian lemma, lemma C, concerning double numerical series. If one attempts to prove an analogous result for the multiplier $\{M_{mn}^{(3)}\}$, one seems to be ineluctably led to another Tauberian "lemma" that is very much like lemma C. However this second "lemma" is false, and we give an example consisting of a concrete numerical double series to show this. Thus we leave as a very delicate open question the following.

PROBLEM. — Does there exist a double series

$$s = \sum c_{mn} e^{i(mx+ny)}$$

and a set E of positive measure so that s converges UR on E, but

$$\sum \frac{m}{\sqrt{m^2+n^2}} c_{mn} e^{i(mx+ny)}$$

diverges UR at all points of E?

The method of proof used to establish our generalized Plessner's theorem (theorem 1) requires getting a lot of quantitative local control over the size of the Poisson integral u of a double series that is UR convergent. More concretely, if the maximal partial sum function associated with a double trigonometric series is finite on a set, then a. e. on that set the $g_\lambda^*(u)$ functions are finite if λ is big enough (theorem 2), the non-tangential maximal functions $N_\alpha(u)$ grow in a measured way as their apex angles open up (theorem 3), and the area integral functions also grow in a measured way as their apex angles open up (theorem 4). These 3 theorems are applied to control the Poisson integrals associated with the Riesz conjugate and the double Riesz conjugate (*see* Lemma B). Theorems 2, 3 and 4 should have many other applications and the reader may well find them the most interesting part of this paper.

2. Notation

We will be studying a double trigonometric series

$$s(\mathbf{x})=\sum_{\mathbb{Z}^2} c_{\mathbf{n}}\, e^{i\mathbf{n}.\mathbf{x}},$$

its first Riesz conjugate

$$s_1(\mathbf{x})=\sum'_{\mathbb{Z}^2} \frac{m}{\sqrt{m^2+n^2}} c_{\mathbf{n}}\, e^{i\mathbf{n}.\mathbf{x}},$$

and its double Riesz conjugate

$$s_{12}(\mathbf{x})=\sum'_{\mathbb{Z}^2} \frac{mn}{m^2+n^2} c_{\mathbf{n}}\, e^{i\mathbf{n}.\mathbf{x}}.$$

Here $\mathbf{x}=(x, y)\in T^2=[0,2\pi]\times[0,2\pi]$; $\mathbf{n}=(m, n)\in\mathbb{Z}^2$, where $\mathbb{Z}^2$ is all ordered pairs of integers; the prime ($'$) means that the (0,0) term is to be omitted; and $\mathbf{n}.\mathbf{x}=mx+ny$. We use the norms

$$|\mathbf{n}|=\sqrt{n^2+m^2} \qquad \text{and} \qquad \|\mathbf{n}\|=\min\{|m|, |n|\}.$$

Write $\mathbf{c}$ for (c, c). If

$$\mathbf{n}=(n_1, n_2) \qquad \text{and} \qquad \mathbf{m}=(m_1, m_2)$$

we write $\mathbf{n}>\mathbf{m}$ for $n_1>m_1$ and $n_2>m_2$, while $\|\mathbf{n}\|>k$ means $n_1>k$ and $n_2>k$. The double sequence $\{t_{\mathbf{n}}\}$ converges *UR* (converges unrestrictedly rectangularly) to t if for every $\varepsilon>0$ there exists $\mathbf{M}=\mathbf{M}(\varepsilon)$ such that $\mathbf{n}>\mathbf{M}$ implies $|t_{\mathbf{n}}-t|<\varepsilon$.

Associated to the series $s(\mathbf{x})$ will be its partial sums $s_{\mathbf{n}}(\mathbf{x})=\sum_{\mathbf{0}\leqslant\mathbf{m}\leqslant\mathbf{n}} c_{\mathbf{m}}\, e^{i\mathbf{m}.\mathbf{x}}$ and its maximal partial sum function

$$s^*(\mathbf{x})=\sup_{\mathbf{n}\geqslant\mathbf{0}} |s_{\mathbf{n}}(\mathbf{x})|.$$

On the bidisc

$$D=\{\mathbf{z}=(z, w)\,|\,|z|<1, |w|<1\}=D_1\times D_2,$$

is defined a formally biharmonic extension of $s(\mathbf{x})$. This extension will be called the Poisson integral of s and is given by the equation

$$u(\mathbf{z})=\sum c_{\mathbf{n}}\, e^{i(m\theta+n\varphi)} r^{|m|} s^{|n|}, \quad \text{where} \quad z=re^{i\theta} \quad \text{and} \quad w=se^{i\varphi}.$$

The Poisson integrals u_1 and u_{12} are derived from s_1 and s_{12} respectively in a similar manner.

It will often suffice to assume $s(\mathbf{x})$ real valued in which case we may write

$$s(\mathbf{x})=\sum_{\mathbf{n}\geqslant\mathbf{0}} A_{\mathbf{n}}(\mathbf{x}), \qquad s_1(\mathbf{x})=i\sum_{\mathbf{n}\geqslant\mathbf{0}} \frac{m}{|\mathbf{n}|} B_{\mathbf{n}}(\mathbf{x}),$$

$$s_{12}(\mathbf{x})=-\sum_{\mathbf{n}\geqslant\mathbf{0}} \frac{mn}{|\mathbf{n}|^2} D_{\mathbf{n}}(\mathbf{x}),$$

where for $(m, n)\neq(0,0)$,

$$A_{\mathbf{n}}(\mathbf{x})=a_{\mathbf{n}}\cos mx\cos ny+b_{\mathbf{n}}\sin mx\cos ny +c_{\mathbf{n}}\cos mx\sin ny+d_{\mathbf{n}}\sin mx\sin ny,$$

$$B_{\mathbf{n}}(\mathbf{x})=-b_{\mathbf{n}}\cos mx\cos ny+a_{\mathbf{n}}\sin mx\cos ny -d_{\mathbf{n}}\cos mx\sin ny+c_{\mathbf{n}}\sin mx\sin ny,$$

$$C_{\mathbf{n}}(\mathbf{x})=-c_{\mathbf{n}}\cos mx\cos ny-d_{\mathbf{n}}\sin mx\cos ny +a_{\mathbf{n}}\cos mx\sin ny+b_{\mathbf{n}}\sin mx\sin ny,$$

$$D_{\mathbf{n}}(\mathbf{x})=d_{\mathbf{n}}\cos mx\cos ny-c_{\mathbf{n}}\sin mx\cos ny -b_{\mathbf{n}}\cos mx\sin ny+a_{\mathbf{n}}\sin mx\sin ny.$$

(interpret $\cos 0x$ to be 1, $\sin 0x$ to be 0, etc.); A_{00} is a constant; and $B_{00}=C_{00}=D_{00}=0$. In this case

$$u(\mathbf{z})=\sum_{\mathbf{n}\geqslant\mathbf{0}} A_{\mathbf{n}}(\mathbf{x})\, r^m s^n, \qquad u_1(\mathbf{z})=i\sum_{\mathbf{n}\geqslant\mathbf{0}} \frac{m}{|\mathbf{n}|} B_{\mathbf{n}}(\mathbf{x})\, r^m s^n$$

and

$$u_{12}(\mathbf{z})=-\sum_{\mathbf{n}\geqslant\mathbf{0}} \frac{mn}{|\mathbf{n}|^2} D_{\mathbf{n}}(\mathbf{x})\, r^m s^n.$$

We will make use of a sequence of Stolz domains (denoted $\Gamma_{\mathbf{n}}(\mathbf{x})$) and also a sequence of "Stolz-like" domains (denoted $\Delta_{\mathbf{n}}(\mathbf{x})$). For any $0<p<1$ let C_p denote the circumference $|z|=p$. Form the open region bounded by the two tangents from $z=1$ to C_p and by the more distant arc of C_p between the points of contact. If $p=\cos\alpha$, $\alpha=\alpha_m=\pi/4\ 1/2^m$, call this open region the m-th Stolz domain Γ_m, $m=0, 1, 2\ldots$ Note that α is also the angle between the tangent to the unit circle at $z=1$ and

Γ_m. The law of sines applied to the triangle with vertices 0, 1, and $z=re^{i\theta}$ shows that

$$\Gamma_m=\{z=re^{i\theta}\,|\,|z|<\cos\alpha\}$$
$$\cup\left\{z=re^{i\theta}\,|\,|\theta|<\alpha \quad \text{and} \quad r<\frac{\cos\alpha}{\cos(\alpha-|\theta|)}\right\}.$$

By $\Gamma_m(x)$ we mean the domain Γ_m rotated through an angle x around $z=0$. Finally $\Gamma_{\mathbf{n}}(\mathbf{x})=\Gamma_m(x)\times\Gamma_n(y)$.

If $z=re^{i\theta}$ we will systematically write $\delta=1-r$. Let Δ_m be the open set which is the convex hull (denoted as CH) of the disjoint open sets

$$\{z=re^{i\theta}\,|\,r<1-\pi 2^{-m}\}$$

and

$$\{z=re^{i\theta}\,|\,1-\pi 2^{-m}<r<1-|\theta|2^{-m}\}.$$

It is clear that

$$\Delta_m=CH(\{z=re^{i\theta}\,|\,\pi 2^{-m}<\delta\}\cup\{z=re^{i\theta}\,|\,|\theta|2^{-m}<\delta<\pi 2^{-m}\}).$$

This domain is more suited to polar coordinates and is equivalent to Γ_m in the sense that $\Delta_m\subseteq\Gamma_m\subseteq\Delta_{2m+4}$. These containments are proved in the appendix. It is immediate that the sequences $\{\Gamma_m\}$ and $\{\Delta_m\}$ are increasing, i. e. $m'>m$ implies $\Gamma_{m'}\supseteq\Gamma_m$ and $\Delta_{m'}\supseteq\Delta_m$. Define $\Delta_{\mathbf{n}}(\mathbf{x})$ as a product of two rotated Δ's as was done for $\Gamma_{\mathbf{n}}(\mathbf{x})$. *Let*

$$\Omega=\left\{z\,|\,\frac{1}{2}<|z|<1\right\} \quad \text{and} \quad \Omega=\Omega\times\Omega.$$

Associated to a biharmonic function u on the bidisc D will be several $[0, \infty]$-valued functions defined on its distinguished boundary T^2. For $\mathbf{n}$ fixed we have the non-tangential maximal function

$$N_{\mathbf{n}}(u)(\mathbf{x})=\sup_{\mathbf{z}\in\Gamma_{\mathbf{n}}(\mathbf{x})}|u(\mathbf{z})|;$$

and the area function

$$[A_{\mathbf{n}}(u)(\mathbf{x})]^2=\int_{\Gamma_{\mathbf{n}}(\mathbf{x})\cap\Omega}|\nabla_1\nabla_2 u(\mathbf{z})|^2\,d\mathbf{z},$$

where

$$|\nabla_1\nabla_2 u(\mathbf{z})|^2=\left(\frac{\partial^2 u}{\partial x_1\,\partial x_2}\right)^2+\left(\frac{\partial^2 u}{\partial x_1\,\partial y_2}\right)^2+\left(\frac{\partial^2 u}{\partial y_1\,\partial x_2}\right)^2+\left(\frac{\partial^2 u}{\partial y_1\,\partial y_2}\right)^2,$$
$$\mathbf{z}=(x_1+iy_1,\, x_2+iy_2) \quad \text{and} \quad d\mathbf{z}=dx_1\,dy_1\,dx_2\,dy_2;$$

and for each fixed $\lambda>0$ there is the function

$$[g_\lambda^*(u)(x, y)]^2=\int_\Omega\left(\frac{\delta\varepsilon}{(\delta+|\theta|)(\varepsilon+|\varphi|)}\right)^\lambda|\nabla_1\nabla_2 u(re^{i(x-\theta)}, se^{i(y-\varphi)})|^2\,d\mathbf{z}$$

where

$$\mathbf{z}=(re^{i\theta}, se^{i\varphi}), \quad \delta=1-r \quad \text{and} \quad \varepsilon=1-s.$$

We will use the summation by parts formula

$$(0)\quad \sum_{j=m}^M\sum_{k=n}^N c_{jk}\lambda_{jk}=\sum_{j=m}^{M-1}\sum_{k=n}^{N-1}\Delta_{jk}^{11}s_{mn}^{jk}+\sum_{j=m}^{M-1}\Delta_{jN}^{10}s_{mn}^{jN}+\sum_{k=n}^{N-1}\Delta_{Mk}^{01}s_{mn}^{Mk}+\lambda_{MN}s_{mn}^{MN}$$

where

$$s_{mn}^{jk}=\sum_{\mu=m}^j\sum_{\nu=n}^k c_{\mu\nu},$$
$$\Delta_{jN}^{10}=\lambda_{jN}-\lambda_{(j+1)N},$$
$$\Delta_{Mk}^{01}=\lambda_{Mk}-\lambda_{M(k+1)},$$

and

$$\Delta_{jk}^{11}=\lambda_{(j+1)(k+1)}-\lambda_{(j+1)k}-\lambda_{j(k+1)}+\lambda_{jk}.$$

The usual generic constant C will be used. Its exact value can change from line to line, but it is always an absolute constant unless otherwise indicated.

All sets will Lebesgue measurable.

3. Results

The main result is:

THEOREM 1. — *If the double trigonometric series $s(\mathbf{x})$ converges UR on a set E, then its double Riesz conjugate $s_{12}(\mathbf{x})$ converges UR almost everywhere on E.*

To prove this we will have to control some $g_\lambda^*(u)(\mathbf{x})$ where u is the Poisson integral of s.

THEOREM 2. — *If the maximal partial sum function $s^*(\mathbf{x})$ of s is finite on a set E, then $g_{28}^*(u)(\mathbf{x})$ is also finite a. e. on E.*

Since g_λ^* is essentially a weighted sum of area functions the proof of theorem 2 rests on the following two results.

THEOREM 3. — *Assume that the maximal partial sum function $s^*(\mathbf{x})$ of s is finite on a set E and let $\varepsilon>0$ be given. Then there is a set $F\subseteq E$, $|E\setminus F|<\varepsilon$ and a constant $M>0$ so that for all $\mathbf{n}\geqslant\mathbf{0}$ and all $\mathbf{x}\in F$,*

$$N_{\mathbf{n}}(u)(\mathbf{x})\leqslant M\,2^{4\,(m+n)}, \tag{1}$$

where u is the Poisson integral of s.

THEOREM 4. — *Let u be a biharmonic function satisfying*

$$N_{\mathbf{n}}(u)(\mathbf{x})\leqslant\lambda_{\mathbf{n}}, \tag{2}$$

for all $\mathbf{x}\in F$ and all $\mathbf{n}\geqslant\mathbf{0}$, and let $\varepsilon>0$ be given. Then there is a $G\subseteq F$, $|F\setminus G|<\varepsilon$, and a constant $C>0$ such that for all $\mathbf{n}\geqslant\mathbf{0}$,

$$\int_G [A_{\mathbf{n}}(\mathbf{x})]^2\,dx\leqslant C\,\lambda_{2\,\mathbf{n}+\mathbf{4}}^2\,2^{10\,(m+n)}. \tag{3}$$

The proof of theorem 3 follows from a Tauberian lemma which is essentially due to PETERSON and WELLAND [11].

LEMMA A. — *Let $s(\mathbf{x})=\sum A_{\mathbf{n}}(\mathbf{x})$ be a real-valued double trigonometric series. Let $s^*(\mathbf{x})$ be finite for $\mathbf{x}\in E$ and let $\varepsilon>0$ be given. Then there is a set $F\subseteq E$ with $|E\setminus F|<\varepsilon$ and a constant K such that for all $\mathbf{x}\in F$ and all $\mathbf{N}=(M,\,N)\geqslant\mathbf{0}$*

$$\left|\frac{1}{M}\sum\nolimits_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{N}} mB_{\mathbf{n}}(\mathbf{x})\right|\leqslant K, \tag{4}$$

$$\left|\frac{1}{N}\sum\nolimits_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{N}} nC_{\mathbf{n}}(\mathbf{x})\right|\leqslant K, \tag{5}$$

$$\left|\frac{1}{MN}\sum\nolimits_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{N}} mnD_{\mathbf{n}}(\mathbf{x})\right|\leqslant K. \tag{6}$$

If further, s converges at each point of E, then we can arrange to have equations (4), (5) and (6) hold and also have an integer N_0 so that if $\|\mathbf{N}\|\geqslant N_0$, for all $\mathbf{x}\in F$

$$\left|\frac{1}{M}\sum\nolimits_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{N}} m\,B_{\mathbf{n}}(\mathbf{x})\right|<\varepsilon, \tag{7}$$

$$\left|\frac{1}{MN}\sum\nolimits_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{N}} mn\,D_{\mathbf{n}}(\mathbf{x})\right|<\varepsilon. \tag{8}$$

Kent Merryfield has proven what amounts to theorem 4 for $\mathbb{R}^n$ instead of $\mathbf{T}^n$ with his biharmonic function living on a product of upper half planes instead of on a product of disks [9]. We simply translate his proof, line by line, to our setting and keep track of constants somewhat more closely than he does. Work in this area has also been done by MALLIAVAN and MALLIAVAN [7] and OKADA [10].

The next lemma is very much like a result of GUNDY and STEIN [5].

LEMMA B. — *If $g_{28}^*(u)$ is finite on a set, $N_{\mathbf{0}}(u_{12})$ is also finite almost everywhere on that set.*

The last link in the chain of results leading to theorem 1 is our Tauberian lemma.

LEMMA C. — *If the double series $\sum_{\mathbf{n}\geqslant\mathbf{0}} a_{\mathbf{n}}$ is rectangularly unrestrictedly Abel summable, i. e. if there is a number a such that*

$$\lim\nolimits_{(r,\,s)\to(1^-,\,1^-)}\sum\nolimits_{\mathbf{n}\geqslant\mathbf{0}} a_{\mathbf{n}}\,r^m\,s^n=a, \tag{9}$$

and if the Tauberian conditions

$$\operatorname{limit}_{\|\mathbf{n}\|\to\infty}\varepsilon_{\mathbf{n}}=0 \tag{10}$$

$$\text{all } \varepsilon_{\mathbf{n}} \text{ bounded}, \tag{11}$$

(where $\varepsilon_{\mathbf{N}}=1/MN\sum_{\mathbf{0}\leqslant\mathbf{n}\leqslant\mathbf{N}}|\mathbf{n}|^2\,a_n$) hold, then $\Sigma\,\mathbf{a}_n$ is UR convergent to a.

Here is a heuristic "diagram" of the proof of theorem 1.

$$\begin{array}{ccc} u & \overset{2}{\rightarrow} & u_{12} \\ {\scriptstyle 1}\uparrow & & \downarrow{\scriptstyle 3} \\ s & & s_{12} \end{array}$$

Arrow 1 refers to theorems 2, 3, 4, all of which deduce good behavior of u from good behavior of s. In other words, these theorems are of an Abelian type. Arrow 2 represents lemma B. Finally arrow 3 represents our Tauberian lemma C.

One would, of course, like to replace s_{12} by s_1 in the conclusion of theorem 1. (In particular one could immediately deduce theorem 1 as stated by an iteration of that result.) The problem with extending our proof to the case of s_1 lies entirely with the analogue of arrow 3. (Kent Merryfield has observed that there is also a minor point involved in extending Lemma B. In the definition of g_λ^*, $|\nabla_1 \nabla_2 u|^2$ must be replaced by $|\nabla_1 \nabla_2 u|^2 + |\nabla_1 u|^2 + |\nabla_2 u|^2$ in order to also control the behavior of $N_0(u_i)$, $i=1, 2$.) By comparing equations (4) and (7) with the corresponding equations (6) and (8) it is clear what the analogue of lemma C must be. Unfortunately this analogue is false. In fact we have the following example.

Example. — There is a numerical series $\Sigma a_{\mathbf{n}}$ which is rectangularly unrestrictedly Abel summable, and UR divergent, even though $\alpha_{\mathbf{n}}$ are bounded and satisfy

$$\lim_{\|\mathbf{n}\| \to \infty} \alpha_{\mathbf{n}} = 0 \qquad \text{where} \quad \alpha_{\mathbf{N}} = \frac{1}{M} \sum_{0 \leq \mathbf{n} \leq \mathbf{N}} |\mathbf{n}| \, a_{\mathbf{n}}.$$

This example shows that we can't get our desired analogue of theorem 1 by the methods of this paper, but is by no means decisive. (For example there could be Tauberian conditions stronger than (4) and (7) lurking in the hypothesis of s converging on a set.) The negative evidence of the example coupled with the contravening positive evidence of the Peterson-Welland result stating that s_1 converges *restrictedly* rectangularly a. e. where s does led us to state our open question concluding the introduction as a problem rather than as a conjecture one way or the other.

4. Proofs

In theorems 1-4 and lemmas A and B the conclusions are all true by default if the hypotheses hold on a set of measure 0, so we will always assume implicity that the given set on which the hypotheses hold has positive measure.

Discussion of Lemma A. — The proof of this result follows the one dimensional ones given for lemmas 4.12 and 4.18 in volume 2 of *Trigonometric Series* [14], pp. 217-219. PETERSON and WELLAND have already sketched the two dimensional argument [11]. For these reasons we will not give the proof here, but will restrict ourselves to some remarks. One uses the notion of stable convergence and identifies such as these.

$$A_{\mathbf{n}}(x+s, y) = A_{\mathbf{n}}(\mathbf{x}) \cos ms - B_{\mathbf{n}}(\mathbf{x}) \sin ms$$

$$\begin{aligned} A_{\mathbf{n}}(x+s, y+t) = A_{\mathbf{n}}(\mathbf{x}) \cos ms \cos nt \\ - B_{\mathbf{n}}(\mathbf{x}) \sin ms \cos nt \\ - C_{\mathbf{n}}(\mathbf{x}) \cos ms \sin nt + D_{\mathbf{n}}(\mathbf{x}) \sin ms \sin nt. \end{aligned}$$

One delicate point in the proof is that from

$$T_{\mathbf{N}}(\mathbf{x}) = \frac{1}{M} \sum_{0 \leq \mathbf{n} \leq \mathbf{N}} m B_{\mathbf{n}}(\mathbf{x}) \to 0 \quad \text{as } \|\mathbf{N}\| \to \infty,$$

it does *not* trivially follow that there is a constant $K(\mathbf{x})$ so that $|T_{\mathbf{N}}(\mathbf{x})| < K(\mathbf{x})$ for all $\mathbf{N}$. (*See* [3], p. 406 for a related example.) However, if $T_{\mathbf{N}}(\mathbf{x}) \to 0$ as $\|\mathbf{N}\| \to \infty$ for all $\mathbf{x} \in E$, then for a. e. $\mathbf{x} \in E$ there is a $K(\mathbf{x})$ so that $|T_{\mathbf{N}}(\mathbf{x})| < K(\mathbf{x})$. This is proved in very much the same way as is lemma 2.3 of [3].

Proof of Theorem 3. — By virtue of lemma A and routine uniformizing arguments we may assume that a large subset F and a constant K have been found so that equations (4)-(6) are valid on F and also that

$$(12) \qquad s^*(\mathbf{x}) < K \quad \text{for all } \mathbf{x} \in F.$$

Let

$$\beta_{\mathbf{N}} = \sum_{0 \leq \mathbf{n} \leq \mathbf{N}} m B_{\mathbf{n}} \qquad \text{and} \qquad \delta_{\mathbf{N}} = \sum_{0 \leq \mathbf{n} \leq \mathbf{N}} mn D_{\mathbf{n}}.$$

Fix $\mathbf{x} = (x, y)$ in F.

A point of

$\Gamma_{\mathbf{n}}(\mathbf{x})$ may be written $(re^{i(x+p)}, se^{i(y+q)}) = (ze^{ix}, we^{iy})$

where

$$(z, w) \in \Gamma_m \times \Gamma_n.$$

Then

$$u(\mathbf{z})=A_0+\sum_{m=1}^{\infty} A_{mo}\cos mp\, r^m-\sum_{n=1}^{\infty} B_{on}\sin nq\, s^n$$
$$+\sum_{m,\,n=1}^{\infty} A_{mn}\cos mp\cos nq\, r^m s^n$$
$$-\sum_{m,\,n=1}^{\infty} B_{mn}\sin mp\cos nq\, r^m s^n$$
$$-\sum_{m,\,n=1}^{\infty} C_{mn}\cos mp\sin nq\, r^m s^n$$
$$+\sum_{m,\,n=1}^{\infty} D_{mn}\sin mp\sin nq\, r^m s^n$$
$$=\mathrm{I}+\mathrm{II}-\mathrm{III}+\mathrm{IV}-\mathrm{V}-\mathrm{VI}+\mathrm{VII}.$$

Each of these terms is estimated in a way which directly generalizes the proof of a lemma in Zygmund's book. (*See* Lemma 4.15, p. 218, [14]). We will give the details for terms V and VII.

For V we start with the identify

$$V=\frac{1}{2}Re\left\{\int_o^p \sum_{m,\,n=1} m\, B_{\mathbf{n}}(\bar{z}^m w^n+z^m w^n)\, dp\right\},$$

where $\bar{z}$ is the conjugate of z, and the identity

$$\sin mp=\int_o^p m\cos mp\, dp \text{ has been used.}$$

Summation by parts (*see* (0)) gives

$$\sum_{m,\,n=1}^{M,\,N} mB_{\mathbf{n}}z^m w^n=\sum_{m,\,n=1}^{M-1,\,N-1}\beta_{\mathbf{n}}z^m w^n(1-z)(1-w)$$
$$+w^N\sum_{m=1}^{M-1}\beta_{mn}z^m(1-z)$$
$$+z^m\sum_{n=1}^{N-1}\beta_{Mn}w^n(1-w)+\beta_{MN}z^M w^N.$$

The inequality (4) and the identity

$$(13)\qquad \sum_{m=1}^{\infty} mr^m=\frac{r}{(1-r)^2},$$

give

$$\lim_{M,\,N\to\infty}\left|w^N\sum_{m=1}^{M-1}\beta_{m\,N}z^m(1-z)\right|\leqslant K|1-z|$$
$$\times\lim_{M,\,N\to\infty}s^N\left(\sum_{m=1}^{M-1}mr^m\right)$$

$$\leqslant\frac{K|1-z|}{(1-r)^2}\lim_{N\to\infty}s^N=0.$$

Similarly the third and fourth terms in the last equation also $\to 0$ as $(M,\,N)\to(\infty,\,\infty)$. Applying this to V, we have

$$V=\frac{1}{2}\mathrm{Re}\left\{\int_0^p\left(\sum_{m,\,n=1}^{\infty}\beta_{mn}(z^m w^n(1-\bar{z})(1-w)+z^m w^n(1-z)(1-w)\right)dp\right\}$$

so by (4) and (13) again, $\int_0^p dp=p$, and $r\leqq 1$,

$$|V|\leqslant\frac{1}{2}K|1-w|(|1-z|+|1-\bar{z}|)\frac{1}{(1-r)^2}\cdot\frac{1}{1-s}|p|$$
$$=K\left(\frac{|1z|}{1-r}\right)\left(\frac{|1-w|}{1-s}\right)\left(\frac{|p|}{1-r}\right)$$
$$\leqslant C\,2^{2m+2n+2m}<C\,2^{4m+4n}$$

since $|1-z|=|1-\bar{z}|$ and geometry facts (37) and (38) apply.

Next the same methods show

$$VII=\frac{1}{2}\mathrm{Re}\left\{\int_0^p\int_0^q\sum_{m,\,n=1}^{\infty} mn\,D_{\mathbf{n}}(z^m w^n+\bar{z}^m w^n)\,dp\,dq\right\},$$
$$=\frac{1}{2}\mathrm{Re}\left\{\int_0^p\int_0^q\sum_{m,\,n=1}^{\infty}\delta_{\mathbf{n}}(1-w)(z^m(1-z)+\bar{z}^m(1-\bar{z}))\,dp\,dq\right\}$$
$$\leqslant\frac{1}{2}K\left(\sum_{m=1}^{\infty}mr^m\right)\left(\sum_{n=1}^{\infty}ns^n\right)|1-w|2|1-z|.|p|.|q|$$
$$\leqslant K\left(\frac{|1-z|}{1-r}\right)\left(\frac{|p|}{1-r}\right)\left(\frac{|1-w|}{1-r}\right)\left(\frac{|q|}{1-s}\right)\leqslant C\,2^{4m+4n}.$$

Proof of Theorem 4. — Let $g=|\nabla_1\nabla_2 u|^2\delta_1\delta_2$. We have to study

$$\iint_{\Gamma_{\mathbf{n}}\cap\Omega} g.$$

By geometric lemma 1 it suffices to study $\iint_{\Delta_{\mathbf{n}}\cap\Omega} g=\iint_D g\,\chi_{\Delta_{\mathbf{n}}\cap\Omega}$.

Following Merryfield we will first deal with an easier integral where the function $\chi_{\Delta_n \cap \Omega}$ is replaced by a smooth approximation called v^2. Lemma 4.1 will be the easier version of Theorem 4.

Its proof will follow the proof of lemma 8 of Merryfield's thesis [8].

LEMMA 4.1. — *Let $b \in C^\infty(\mathbb{R})$ be even, positive, decreasing on $[0, \pi]$ and have support $T := [-\pi, \pi]$. Let*

$$b_a(x) = \frac{1}{a} b\left(\frac{x}{a}\right)$$

and define

$$\varphi_a: \quad T \to \mathbb{R} \quad by \quad \varphi_a(x) = \sum_{n=-\infty}^{\infty} b_a(x + 2\pi n).$$

Let u be biharmonic on D and satisfy

$$N_{2\,\mathbf{n}+4}(u)(\mathbf{x}) \leqslant \lambda_{2\,\mathbf{n}+4} =: \lambda \quad \text{for all } \mathbf{x} \in E \subseteq T.$$

Write

$$v(z, w) = [\varphi_{2^m\delta,\, 2^n\varepsilon}] * \chi_E(x, y) = \iint_{T^2} \varphi_{2^m\delta}(x-s)\, \varphi_{2^n\varepsilon}(y-t)\, \chi_E(s, t)\, ds\, dt.$$

Then

$$\iint_\Omega |\nabla_1 \nabla_2 u|^2 v^2 \delta\varepsilon \leqslant C\, 2^{4m+4n} \lambda^2,$$

where C is an absolute constant.

Proof of lemma 4.1. — We may, without loss of generality, assume that u is also continuous on the closure of Ω. (If necessary, replace u by u_ε where $u_\varepsilon(r, \theta) = u(r - \varepsilon, \theta)$, prove lemma 4.1 for u_ε, and then let $\varepsilon \to 0$.) We begin by establishing a one dimensional result (statement (14) below) which will be used twice, once to prove the one dimensional analogue of 4.1, and a second time to reduce 4.1 to the one dimensional case. Let

$$z = re^{ix}, \qquad \delta = 1 - r, \qquad f \in L^2(T), \qquad v = \varphi_{2^m\delta} * f,$$

$$\rho(x) = \Sigma\,(x + 2\pi\nu)\, b(x + 2\pi\nu) \qquad \text{and} \qquad w = 2^m \rho_{2^m\delta} * f.$$

Note

$$v = \Sigma \hat{f}(\nu)\, \hat{b}(2^m\delta\nu)\, e^{i\nu x} \qquad \text{where} \qquad f = \Sigma \hat{f}(\nu)\, e^{i\nu x}$$

so

$$\nabla v = \frac{\partial v}{\partial r} e_r + \frac{1}{r} \frac{\partial v}{\partial x} e_x = \Sigma \hat{f}(\nu) \left[-2^m \nu\, \hat{b}'(2^m\delta\nu)\, e_r + \frac{i\nu\, \hat{b}(2^m\delta\nu)}{r} e_x \right]$$

where e_r and e_x are unit vectors in the radial and angular directions respectively. Define a vector valued function ψ by

$$\psi = 3\left(\delta \frac{\partial}{\partial r}(\varphi_{2^m\delta}),\ \frac{\delta}{r} \frac{\partial}{\partial x}(\varphi_{2^m\delta}),\ 2^m \rho_{2^m\delta}\right)$$

so that

$$|\psi * f|^2 = 9\,|\delta \nabla v|^2 + 9\,|w|^2.$$

Then if $u(z)$ is continuous on $\bar{\Omega}$ and harmonic on Ω,

$$(14) \quad \int_\Omega |\nabla u|^2 v^2 \delta \leqslant \int_\Omega |u|^2 |\psi * f|^2 \delta^{-1} + 2\int_\Omega u^2 v^2 + \int_{\delta=0} |fu|^2 + C \int_{\delta=1/2} |uv|^2.$$

To prove (14), apply Green's theorem in the form

$$\int_\Omega \nabla u \cdot g = -\int_\Omega u \operatorname{div} g + \int_{\partial\Omega} u\,(g \cdot \mathbf{n})$$

where

$$g := (\nabla u)(\varphi_{2^m\delta} * f)^2 = (\nabla u)\, v^2.$$

We get

$$I := \int_\Omega |\nabla u|^2 v^2 \delta = -2\int_\Omega uv\, \nabla u\, \nabla v\, \delta + \int_\Omega uv^2 \frac{\partial u}{\partial r} + \int_{|z|=1/2} -\frac{1}{2} v^2 \frac{\partial u}{\partial r} u =: \mathbf{A} + \mathbf{B} + \mathbf{C}.$$

Note that $\delta = 0$ on $|z| = 1$ eliminated another term like $\mathbf{C}$.

By the Peter-Paul inequality ($2ab \leqslant 1/4\, a^2 + 4\, b^2$),

$$\mathbf{A} \leqslant 2\int_\Omega (\delta^{1/2}\, v\, \nabla u) \,.\, (u\, \delta^{1/2}\, \nabla v) \leqslant \frac{1}{4}\int_\Omega |\nabla u|^2 v^2 \delta + 4\int_\Omega |u|^2 |\delta \nabla v|^2 \frac{1}{\delta}.$$

To treat $\mathbf{B}$ we need a further integration by parts:

$$\begin{aligned}\mathbf{B} &= \frac{1}{2}\int_0^{2\pi}\left(\int_{1/2}^1 (rv^2)\frac{\partial}{\partial r}(u^2)\, dr\right) dx \\ &= \frac{1}{2}\int_0^{2\pi}\left(ru^2 v^2\Big|_{1/2}^1 - \int_{1/2}^1 u^2\left(v^2 + 2rv\frac{\partial v}{\partial r}\right) dr\right) dx \\ &= \frac{1}{2}\int_0^{2\pi} f^2(x)\, u^2(x, 0)\, dx - \frac{1}{4}\int_0^{2\pi} v^2\left(x, \frac{1}{2}\right) dx \\ &\qquad - \frac{1}{2}\int_\Omega \frac{u^2 v^2}{r} - \int_\Omega u^2 v \frac{\partial v}{\partial r} =: \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3 + \mathbf{B}_4.\end{aligned}$$

For $\mathbf{B}_1$, note that $v(x, 0) = f(x)$ a.e. and u is assumed continuous at $\delta = 0$.

Since

$$r \geqslant \frac{1}{2} \quad \text{on } \Omega, \qquad \mathbf{B}_3 < \int_\Omega u^2 v^2.$$

We must still integrate $\mathbf{B}_4$ by parts again. However this integration by parts will be done in the angular direction by means of the identity

$$\frac{\partial}{\partial \delta}(\varphi_{2^m \delta} * f) = -\frac{\partial}{\partial x}(2^m \rho_{2^m \delta} * f).$$

Then

$$\begin{aligned}\mathbf{B}_4 &= \int_\Omega u^2 v \frac{\partial}{\partial x}(2^m \rho_{2^m \delta} * f) \\ &= \int_{1/2}^1 \left(u^2 vw\Big|_0^{2\pi} - \int_0^{2\pi} w \frac{\partial}{\partial x}(u^2 v)\right) r dr \\ &= -2\int_\Omega u\frac{\partial u}{\partial x} vw - \int_\Omega u^2 w \frac{\partial v}{\partial x} =: \mathbf{B}_{41} + \mathbf{B}_{42},\end{aligned}$$

since the integrated terms vanish by periodicity.

Apply the Schwarz inequality and then the Peter-Paul inequality to $\mathbf{B}_{41}$ to get

$$\begin{aligned}|\mathbf{B}_{41}| &\leqslant 2\left(\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 v^2 \delta\right)^{1/2}\left(\int_\Omega u^2 w^2 \delta^{-1}\right)^{1/2} \\ &\leqslant \frac{1}{4}\int_\Omega |\nabla u|^2 v^2 \delta + 4\int_\Omega |u|^2 w^2 \delta^{-1},\end{aligned}$$

since

$$\left|\frac{\partial u}{\partial x}\right|^2 \leqslant \left|\frac{1}{r}\frac{\partial u}{\partial x}\right|^2 \leqslant |\nabla u|^2.$$

Apply the Schwarz inequality and $|ab| \leqslant 1/2(a^2 + b^2)$ to $\mathbf{B}_{42}$.

$$\begin{aligned}|\mathbf{B}_{42}| &\leqslant \left(\int_\Omega |u|^2 \left|\delta\frac{\partial v}{\partial x}\right|^2 \delta^{-1}\right)^{1/2}\left(\int_\Omega |u|^2 |w|^2 \delta^{-1}\right)^{1/2} \\ &\leqslant \frac{1}{2}\int_\Omega |u|^2 |\delta \nabla v|^2 \delta^{-1} + \frac{1}{2}\int_\Omega |u|^2 |w|^2 \delta^{-1}.\end{aligned}$$

We turn our attention to $\mathbf{C}$. Even if $m=0$, $\Gamma_{2.0+4} = \Gamma_4$ contains an origin centered circle of radius $\cos((\pi/4)\,.\,(1/2^4)) > .99$ so for *any* x and $m \geqslant 0$, the entire circle $|z| = 1/2$ is contained in $\Gamma_{2m+4}(x)$. In fact if $|z| = 1/2$, there is a circle of radius at least $.99 - .75 = .24$ about z within each $\Gamma_{2m+4}(x)$.

By a standard argument involving the mean value property of harmonic functions (*see* [12], p. 275, c.3), it follows that $|\partial u/\partial r| < C|u|$ on $|z| = 1/2$ for some constant C. Hence

$$|\mathbf{C}| \leqslant \frac{1}{2}\int_{|z|=1/2} |u|(C|u|)|v|^2 < C\int_{|z|=1/2} |u|^2 |v|^2.$$

Combining our estimates for $\mathbf{A}$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_{41}$, $\mathbf{B}_{42}$ and $\mathbf{C}$,

$$\begin{aligned}I &\leqslant \frac{1}{2} I + \frac{1}{2}\int_0^{2\pi} |f(x)|^2 |u(x, 0)|^2 dx \\ &\qquad + \frac{1}{4}\int_0^{2\pi} v^2\left(x, \frac{1}{2}\right) u^2\left(x, \frac{1}{2}\right) dx + \int_\Omega u^2 v^2\end{aligned}$$

$$+\frac{9}{2}\int_{\Omega} u^2(|\delta\nabla u|^2+|w|^2)\frac{1}{\delta}+\frac{1}{2}\int_{|z|=1/2} v^2\frac{\partial u}{\partial x}u.$$

Recall the definition of ψ, transpose $1/2I$ to the left hand side, and multiply by 2 to get equation (14).

The one dimensional version of lemma 4.1 is the following.

(15) Let u be harmonic on Ω and continuous on its closure, and let $f\in L^2(T^1)$. If $N_{2m+4}(u)\leqslant\lambda$ for $x\in\operatorname{supp}(f)$, then there is an absolute constant C such that

$$\int_{\Omega}|\nabla u|^2|\varphi_{2^m\delta}*f|^2\delta\leqslant C\,4^m\lambda^2\int_0^{2\pi}|f(\mathbf{x})|^2\,dx.$$

Let $F:=\operatorname{supp}(f)\subseteq T^2$, and let $S=\bigcup_{x\in F}\Delta_m(x)$ be the associated "saw-tooth region". Then

$$\operatorname{supp} v(x,\delta)\subseteq S. \tag{16}$$

This is trivial if $2^m\delta\geqslant\pi$ since then each Δ_m contains the entire circle of radius $1-\delta$, while if

$$2^m\delta<\pi,\ v(x,\delta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2^m\delta}b\left(\frac{x-t}{2^m\delta}\right)f(t)\,dt$$

and support $b=[-\pi,\pi]$.

Since $\Delta_m(x)\subseteq\Gamma_{2m+4}(x)$ by geometric lemma 1 we always have

$$|u(z)|\leqslant N_{2m+4}(u)(x)\quad\text{for all } z \text{ in } \Delta_m(x). \tag{17}$$

From the definitions of ψ and ρ and our expansion of ∇v, Plancherel's formula gives

$$\int_0^{2\pi}|\psi*f|^2\,dx=2\pi\sum_v|\hat\psi(\nu)|^2|\hat f(\nu)|^2$$

$$=2\pi\cdot 9\sum_v\left\{(2^m\nu\delta)^2|\hat b'(2^m\nu\delta)|^2+\frac{(\nu\delta)^2|\hat b(2^m\delta\nu)|^2}{r^2}\right.$$

$$\left.+2^m\hat\rho(2^m\delta\nu)|^2\right\}|\hat f(\nu)|^2.$$

$$=:18\pi\sum_\nu\{A(\nu,\delta,m,b,\rho)\}|\hat f(\nu)|^2.$$

Now multiply by δ^{-1} and by the Jacobian r. Use $1/r<2$ and $r<1$ to eliminate any powers of r. Integrate in δ from 0 to 1/2 (which amounts to integrating in ν from 1/2 to 1) and note that the substitution $t=2^m\nu\delta$ shows

$$\int_0^{2\pi}A(\nu,\delta,m,b,\rho)\frac{d\delta}{\delta}<2\int_{-\infty}^{\infty}\left(|t^2\hat b'(t)|^2+\frac{1}{4^m}|t\hat b(t)|^2\right)\frac{dt}{t}.$$

$\left(\text{Since } \hat\rho(0)=\int_0^{2\pi}\rho(x)\,dx=0 \text{ and } \hat\rho \text{ is } C^\infty,\ \hat\rho(t)=0(t) \text{ near } t=0 \text{ so that this last integral is dominated by } 4^m C.\right)$ We conclude that

$$\int_{\Omega}|\psi*f|^2\delta^{-1}<C\,4^m. \tag{18}$$

We are now ready to prove (15). From (14) and the assumptions in (15),

$$\int_{\Omega}|\nabla u|^2v^2\delta\leqslant\lambda^2\int_{\Omega}|\psi*f|^2\delta^{-1} \tag{19}$$

$$+2\lambda^2\int_{\Omega}v^2+\lambda^2\int_0^{2\pi}|f|^2+C\lambda^2\int_{|z|=1/2}|v|^2,$$

where the domination $|u|\leqslant\lambda$ is justified by relation (16). Young's inequality ($\|\varphi*f\|_2\leqslant\|\varphi\|_1\|f\|_2$) for each δ, $0<\delta\leqslant 1/2$, gives

$$\int_{|z|=\delta}|v|^2\leqslant\left(\int(\varphi_{2^m\delta})\right)^2\int_0^{2\pi}|f(x)|^2\,dx=\|f\|_2^2.$$

Set $\delta=1/2$ to estimate $\int_{|z|=1}v^2$. Multiply by the Jacobian r and integrate from 1/2 to 1 to get

$$\int_{\Omega}v^2<\|f\|_2\int_{1/2}^1 r\,dr=1/4\|f\|_2.$$

Inequalities (18) and (19) then imply the validity of statement (15).

We now pass to the proof of lemma 4.1 itself. For our original u on Ω, define $u_w(z):=u(z, w)$. Then for each w, u_w is harmonic and

(20) $$N_n(N_m(u_w(\mathbf{x})(y)) = N_{\mathbf{n}}(x, y).$$

In particular, if $(x, y, \delta, \varepsilon)\in \operatorname{supp}(\varphi_{2^m\delta,\, 2^n\varepsilon} * f)$, then

(21) $$N_m(u_{y,\,\varepsilon}(x)) \leqslant \inf_{|y-y|\leqslant 2^n\varepsilon} N_{\mathbf{n}}(u)(x, y)\leqslant\lambda.$$

We also define the following vector-valued function:

$$g_w(x)=\int_0^{2\pi}\psi_{2^n\varepsilon}(y-y)\, f(x, y)\,dy,$$

where $\psi_{2^n\varepsilon}(y)$ is an $\mathbb{R}^3$-valued function defined in the same way as $\psi_{2^m\delta}(x)$ was. In exactly the same manner as inequality (18) was derived we have

$$\int_\Omega |g_w(x)|^2\varepsilon^{-1}\,dw < C\,4^n\int_0^{2\pi}|f(x, y)|^2\,dy.$$

But this implies

(22) $$\int_\Omega \|g_w(x)\|^2_{L^2(T)}\varepsilon^{-1}\,dw = \int_0^{2\pi}\left[\int_\Omega |g_w(x)|^2\varepsilon^{-1}\,dw\right]dx \leqslant C\,4^n\|f\|^2_{L^2(T^2)}.$$

In particular, $\|g_w(x)\|^2_{L^2(T^1)}<\infty$ for almost every $w\in\Omega$.

Now equation (14) applied to the second variable gives

$$I:=\int_\Omega |\nabla_1\nabla_2 u|^2(\varphi_{2^m\delta,\,2^n\varepsilon}\bigstar\chi_E)^2\,\delta\varepsilon\,dz\,dw$$
$$=\int_\Omega\left[\int_\Omega|\nabla_2(\nabla_1 u)|^2\,|\varphi_{2^m\delta,\,2^n\varepsilon}\bigstar\chi_E|^2\,\varepsilon\,dw\right]\delta dz$$
$$\leqslant \int_\Omega\left[\int_\Omega |\nabla_1 u|^2\,|\psi_{2^n\varepsilon}\bigstar(\varphi_{2^m\delta}\bigstar\chi_E)|^2\frac{1}{\varepsilon}\,dw\right]\delta dz$$

$$+2\int_\Omega\left[\int_\Omega |\nabla_1 u|^2(\varphi_{2^m\delta,\,2^n\varepsilon}\bigstar\chi_E)^2\,dw\right]\delta dz$$
$$+\int_\Omega\left[\int_{|w|=1}|\nabla_1 u|^2(\varphi_{2^m\delta}\bigstar\chi_E)^2\,dy\right]\delta dz$$
$$+C\int_\Omega\left[\int_{|w|=1/2}|\nabla_1 u|^2(\varphi_{2^m\delta,\,2^n\varepsilon}\bigstar\chi_E)^2\,dy\right]\delta dz$$
$$=:\mathbf{A}+\mathbf{B}+\mathbf{C}+\mathbf{D}.$$

Interchange the z and w integrations in all four integrals. Then by (15) and (21),

$$\mathbf{A}=\int_\Omega\left[\int_\Omega|\nabla_1 u_w|^2(\varphi_{2^m\delta} * \psi_{2^n\varepsilon}\chi_E)^2\,\delta dz\right]\frac{1}{\varepsilon}\,dw$$
$$\leqq \int_\Omega\left[C\lambda^2 4^m\left(\int_{|z|=1}(\psi_{2^n\varepsilon}*\chi_E)^2\right)\right]\frac{1}{\varepsilon}\,dw.$$

Apply (22) with $f=\chi_E$ to get

$$\mathbf{A}\leqslant C\lambda^2 4^m 4^n\int_0^{2\pi}\int_0^{2\pi}(\chi_E(x, y))^2\,dy\,dx<C\lambda^2 4^{m+n}.$$

Next by (15) and (21) again,

$$\mathbf{B}=2\int_\Omega\left[\int_\Omega|\nabla_1 u_w|^2\,|\varphi_{2^m\delta}\bigstar(\varphi_{2^n\varepsilon}\bigstar\chi_E)^2\,\delta dz\right]dw$$
$$\leqslant 2\int_\Omega \|C\,4^m\lambda^2([\varphi_{2^n\varepsilon}*\chi_E)(.,\, w)\|_2^2]\,dw.$$

Again Young's inequality gives

$$\|(\varphi_{2^n\varepsilon}\bigstar\chi_E)(.,\, w)\|_2^2$$
$$\leqslant \int_0^{2\pi}|\chi_E(x, y)|^2\,dx \quad\text{for each } w=(y, \varepsilon)\text{ in }\Omega.$$

Thus

$$\mathbf{B}\leqslant C\,4^m\lambda^2\int_{1/2}^1\left[\int_0^{2\pi}\left(\int_0^{2\pi}|\chi_E(x, y)|^2\,dx\right)dy\right]ds\leqslant C\lambda^2 4^m.$$

Our third integral is by (15) and (21),

$$\mathbf{C}=\int_0^{2\pi}\left[\int_\Omega |\nabla_1 u_{(y,0)}|^2 (\varphi_{2^m\delta} \bigstar \chi_E)^2\, \delta dz\right] dy$$

$$\leqslant \int_0^{2\pi}\left[C\, 4^m \lambda^2 \int_0^{2\pi} \chi_E(x, y)^2\, dx\right] dy < C\lambda^2 4^m.$$

Once again by (15), (21), and Young's inequality

$$\mathbf{D}=\frac{1}{2}\int_0^{2\pi}\left[\int_\Omega |\nabla_1 u_{(y,(1/2))}|^2 (\varphi_{2^{n-1}} \bigstar \chi_E))^2\, \delta dz\right] dy$$

$$\leqslant \int_0^{2\pi} C\, 4^m \lambda^2 \int_0^{2\pi} (\varphi_{2^{n-1}} \bigstar \chi_E(x, y))^2\, dx\, dy \leqslant C\, 4^m \lambda^2.$$

This proves lemma 4.1.

The following lemma follows one of MERRYFIELD [9], Lemma 4.2, tracking his constants line by line.

LEMMA 4.2. — *Let* $\mathbf{x}$ *be a point of strong density for a set* $E \subseteqq T^2$. *Let*

$$E_\eta=\{\mathbf{x}\in E\,|\,(\varphi_{2^m\delta,\, 2^n\varepsilon} \bigstar \chi_E)(x) \geqslant \eta\, 2^{-m-n}$$

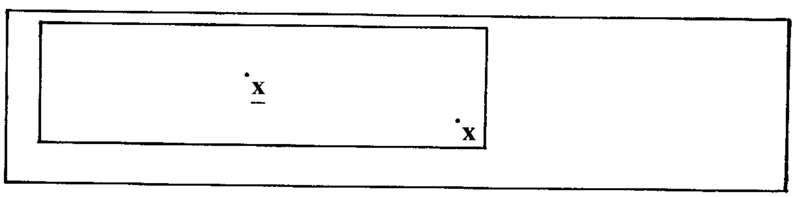
$$\textit{for all } (m, n) \geqslant \mathbf{0} \textit{ and all } (\underline{\mathbf{x}}, \delta, \varepsilon)\in \Delta_{\mathbf{n}}(\mathbf{x})\}.$$

Then $\mathbf{x}\in E_\eta$ *for some* $\eta > 0$.

Proof. — Fix $\mathbf{x}$, a point of strong density of E. Find d, $0 < d < 1$, so that whenever $(\delta, \varepsilon) \leqslant \mathbf{d}$, then

$$\text{(23)}\qquad \frac{|R(x;\, \delta, \varepsilon) \cap E|}{|R(x;\, \delta, \varepsilon)|} > \frac{1}{2} \quad \text{whenever } \mathbf{x}\in R(\mathbf{x};\, \delta, \varepsilon),$$

where $R(\underline{x};\, \delta, \varepsilon)=[x-\delta, x+\delta]\times[y-\varepsilon, y+\varepsilon]$. *To do this simply pick* d so small that $|R(\mathbf{x};\, 2\delta, 2\varepsilon)\cap E|/|R(\mathbf{x};\, 2\delta, 2\varepsilon)| > 7/8$ whenever $(\delta, \varepsilon) \leqslant \mathbf{d}$. Then (23) follows from this picture:



Fix $(\underline{\mathbf{x}}, \delta, \varepsilon)\in\Delta_{\mathbf{n}}(\mathbf{x})$. Then $\mathbf{x}\in R(\underline{\mathbf{x}};\, 2^m\delta, 2^n\varepsilon)$. (Note if $2^m\delta \geqslant \pi$, that $x\in[x-2^m\delta, x+2^m\delta]$ is trivial.) Let

$$(g, h):=(\min\{d, 2^m\delta\}, \min\{d, 2^n\varepsilon\}).$$

Then

$$\text{(24)}\qquad |R(\mathbf{0};\, g, h)|=4gh \geqslant 4d^2\,\frac{2^m\delta}{2^m}\cdot\frac{2^n\varepsilon}{2^n}$$

$$=d^2\, 2^{-m-n}\,|R(\mathbf{0};\, 2^m\delta, 2^n\varepsilon)|;$$

since if $g=d$, note $\delta \leqslant 1$, while if $g=2^m\delta$, note $d/2^m \leqslant d \leqslant 1$, so that

$$g \geqslant d\cdot\delta=(2^m\delta)\left(\frac{d}{2^m}\right).$$

Since $(g, h) \leqslant (2^m\delta, 2^n\varepsilon)$ for some $\mathbf{z}\in T^2$ we have

$$R(\mathbf{z};\, g, h) \subset R(\underline{\mathbf{x}};\, 2^m\delta, 2^n\varepsilon).$$

Combining this with (23) and (24) we get

$$\text{(25)}\qquad |R(\underline{\mathbf{x}};\, 2^m\delta, 2^n\varepsilon)\cap E| \geqslant |R(\mathbf{z};\, g, h)\cap E|$$

$$\geqslant \frac{1}{2}|R(\mathbf{z};\, g, h)| \geqslant \frac{1}{2}d^2\, 2^{-m-n}\,|R(\mathbf{0};\, 2^m\delta, 2^n\varepsilon)|.$$

Next note that

$$\varphi_{2^{m+1}\delta}(u)=\sum\nolimits_v \frac{1}{2^{m+1}\delta}\, b\left(\frac{u}{2^{m+1}\delta}+2m\right)$$

$$\geqslant \frac{1}{2^{m+1}\delta}\, b\left(\frac{u}{2^{m+1}\delta}\right) \geqslant \frac{1}{2^{m+1}\delta}\, b\left(\frac{1}{2}\right)$$

if $|u| \leqslant 2^m\delta$ so that

$$\varphi_{2^{m+1}\delta,\, 2^{n+1}\varepsilon}(u) \geqslant \frac{b^2(1/2)}{|R(\mathbf{0};\, 2^m\delta, 2^n\varepsilon)|}$$

if $u \in R(0;\, 2^m\delta, 2^n\varepsilon)$.

This and inequality (25) imply

$$(\varphi_{2^m\delta,\,2^n\varepsilon} \star \chi_E)(x) = \int_E \varphi_{2^m\delta,\,2^n\varepsilon}(\underline{\mathbf{x}} - \mathbf{z})\,d\mathbf{z}$$

$$\geqslant \int_{R(\mathbf{x};\,2^m\delta,\,2^n\varepsilon)\cap E} \varphi_{2^m\delta,\,2^n\varepsilon}(\underline{\mathbf{x}} - \mathbf{z})\,d\mathbf{z}$$

$$\geqslant (\inf \varphi_{z \in R(\underline{\mathbf{x}};\,2^m d,\,2^n\varepsilon)}) \cdot |R(\underline{x};\,2^m\delta,\,2^n\varepsilon) \cap E|$$

$$\geqslant \frac{1}{2^m 2^n} \cdot \left(\frac{d^2 \cdot b^2(1/2)}{2}\right).$$

This proves lemma 4.2 with $\eta = d^2 b^2(1/2)/2$.

We can now prove theorem 4. Almost every point of F is a point of strong density [6]. Hence by lemma 4.2, $|F_\eta| \nearrow |F|$ as $\eta \searrow 0$, so by picking η sufficiently small we have $|F \setminus F_\eta| < \varepsilon$. Fixing such an η and setting $G = F \cap F_\eta$ it suffices to show that

$$\text{(26)} \qquad \int_G [A_\mathbf{n}(\mathbf{x})]^2\,d\mathbf{x} \leqslant C\lambda^2_{2\,\mathbf{n}+\mathbf{4}}\, 2^{4(m+n)}.$$

Working from the definitions, using geometric lemma 1 and Fubini's theorem we have with $\mathbf{p} = 2\,\mathbf{n} + \mathbf{4} = (p, q)$,

$$\int_G [A_\mathbf{n}(\mathbf{x})]^2\,d\mathbf{x} = \int_G \left(\int_{\Gamma_\mathbf{n}(\mathbf{x})\cap\Omega} |\nabla_1 \nabla_2 u|^2\,d\mathbf{z}\right) d\mathbf{x}$$

$$\leqslant \int_G \left(\int_{\Delta_\mathbf{p}(\mathbf{x})\cap\Omega} |\nabla_1 \nabla_2 u|^2\,d\mathbf{z}\right) d\mathbf{x}$$

$$= \int_{T^2} \chi_G(\boldsymbol{\theta}) \left(\int_\Omega |\nabla_1 \nabla_2 u(\mathbf{z})|^2\, \chi^{R(\mathbf{x};\,2^p\delta,\,2^q\varepsilon)}(\mathbf{z})\,d\mathbf{z}\right) d\theta$$

$$= \int_\Omega |\nabla_1 \nabla_2 u(\mathbf{z})|^2 \left\{\int_{T^2} \chi_G(\boldsymbol{\theta})\, \chi^{R(\theta;\,2^p\delta,\,2^q\varepsilon)}(\mathbf{z})\,d\theta\right\} d\mathbf{z}$$

$$\leqslant 4 \cdot 2^{p+q} \int_{S_p} |\nabla_1 \nabla_2 u(r,\, s;\, \delta,\, \varepsilon)|^2\,\delta\varepsilon\,d\mathbf{z},$$

where $S_P := (\bigcup_{\theta \in G} \Delta_P(\theta)) \cap \Omega$ since the quantity in curly brackets is bounded by $|R(\mathbf{0};\, 2^p\delta,\, 2^q\varepsilon)|$ and is zero whenever $\mathbf{z} \notin S_p$.

Now apply first lemma 4.2 and then lemma 4.1 to this last integral. We have

$$\int_G [A_\mathbf{n}(\mathbf{x})]^2\,d\mathbf{x} \leqslant 4 \cdot 2^{p+q}$$

$$\times \int_{S_\mathbf{p}} |\nabla_1 \nabla_2 u|^2 \left[\frac{2^{m+n}}{\eta} (\varphi_{2^m\delta,\,2^n\varepsilon} \star \chi_E)(\mathbf{x})\right]^2 \delta\varepsilon\,d\mathbf{z}$$

$$\leqslant 4 \cdot 2^8 \cdot 2^{4m+4n} \left(\frac{2^{m+n}}{\eta}\right)^2$$

$$\times \int_\Omega |\nabla_1 \nabla_2 u|^2 ((\varphi_{2^m\delta,\,2^n\varepsilon} \star \chi_E)(\mathbf{x}))^2\,\delta\varepsilon\,d\mathbf{z}$$

$$\leqslant C\,2^{6(m+n)} \cdot 2^{4(m+n)} \lambda^2 = C\,2^{10(m+n)} \lambda^2.$$

This proves theorem 4.

Proof of theorem 2. — It suffices to show that for each $\varepsilon > 0$ there is a subset $F \subset E$ with $g^*_{28}(\mathbf{x}) < \infty$ on F and $|E \setminus F| < \varepsilon$. Let $\varepsilon > 0$ be given. By virtue of theorems 3 and 4 we may assume that an $F \subset E$ has been chosen so that

$$|E \setminus F| < \varepsilon;$$

$$N_\mathbf{n}(u)(\mathbf{x}) \leqslant M\, 2^{4(m+n)} \quad \text{for all } \mathbf{n} \geqslant \mathbf{0} \text{ and } \mathbf{x} \in F;$$

and

$$\int_F [A_\mathbf{n}(\mathbf{x})]^2\,d\mathbf{x} \leqslant C\lambda^2_{2\,\mathbf{n}+\mathbf{4}}\, 2^{10(m+n)} \quad \text{for all } \mathbf{n} \geqslant \mathbf{0},$$

where $\lambda_\mathbf{n} \leqslant M\, 2^{4(m+n)}$. Combining the last 2 inequalities,

$$\int [A_\mathbf{n}(\mathbf{x})]^2\,d\mathbf{x} \leqslant C\,2^{26(m+n)}.$$

But this, together with (40) from geometric lemma 2 gives us

$$\int_F [g^*_{28}(\mathbf{x})]^2\,d\mathbf{x}$$

$$= \int_F \int_\Omega \left(\frac{\delta\varepsilon}{(\delta + |\theta|)(\varepsilon + |\varphi|)}\right)^{28} |\nabla_1 \nabla_2 u(re^{i(x-\theta)},$$

$$se^{i(y-\varphi)})|^2\,d\mathbf{z}\,d\mathbf{x} \leqslant C\int_F\Big[\textstyle\sum_{\mathbf{n}\geqslant \mathbf{0}} 2^{-28(m+n)}$$

$$\times \int_{\Gamma_{\mathbf{n}}(\mathbf{x})\cap\Omega} |\nabla_1\nabla_2 u(re^{i(x-\theta)}, se^{i(y-\varphi)})|^2\,d\mathbf{z}\Big]\,d\mathbf{x}$$

$$= C\textstyle\sum_{\mathbf{n}\geqslant \mathbf{0}} 2^{-28(m+n)}\displaystyle\int_F [A_{\mathbf{n}}(\mathbf{x})]^2\,d\mathbf{x}$$

$$\leqslant C\textstyle\sum_{\mathbf{n}\geqslant \mathbf{0}} 2^{-2(m+n)} = \dfrac{C}{9}.$$

The first inequality follows from writing

$$\Omega = [(\Gamma_0\cap\Omega)\times(\Gamma_0\cap\Omega)] \cup \textstyle\bigcup_{n=1}^{\infty}(\Gamma_0\cap\Omega)\times((\Gamma_n-\Gamma_{n-1})\cap\Omega),$$

$$\cup \textstyle\bigcup_{m=1}^{\infty}((\Gamma_m\setminus\Gamma_{m-1})\cap\Omega)\times(\Gamma_0\cap\Omega)$$

$$\cup \textstyle\bigcup_{m,\,n=1}^{\infty}(\Gamma_m\setminus\Gamma_{m-1})\cap\Omega)\times((\Gamma_n\setminus\Gamma_{n-1})\cap\Omega),$$

bounding $\delta/(\delta+|\theta|)$ by $1=2^0$ on Γ_0 and by 2^{-i} on $\Gamma_i\setminus\Gamma_{i-1}$, and finally by bounding

$$\int_{((\Gamma_m\setminus\Gamma_{m-1})\cap\Omega)\times((\Gamma_n\setminus\Gamma_{n-1})\cap\Omega)}$$

by

$$\int_{(\Gamma_m\cap\Omega)\times(\Gamma_n\cap\Omega)} = \int_{\Gamma_{\mathbf{n}}\cap\Omega} \quad \text{and so forth.}$$

This proves theorem 2.

Remark. — It is clear from the above proof that $g^*_{26+\varepsilon}$ is finite for any $\varepsilon>0$.

Discussion of lemma B. — Let

$$m_1(\mathbf{n}) = \frac{m}{\sqrt{m^2+n^2}} \quad \text{and} \quad m_2(\mathbf{n}) = \frac{mn}{m^2+n^2}.$$

The functions u_1 and u_{12} may be thought of as convolutions over T^2 of u with M_1 and M_2 respectively, where

$$M_1(\mathbf{x}, r, s) = \Sigma\, m_1(\mathbf{n})\, e^{i\mathbf{n}.\mathbf{x}}\, r^{|m|}\, s^{|n|}$$

and

$$M_{12}(\mathbf{x}, r, s) = \Sigma\, m_2(\mathbf{n})\, e^{i\mathbf{n}.\mathbf{x}}\, r^{|m|}\, s^{|n|}.$$

It is easy to prove by induction that for $\boldsymbol{\alpha}=(\alpha, \beta)\geqslant \mathbf{0}$ in $\mathbb{Z}^2$,

$$\left(\frac{\partial}{\partial x}\right)^{\boldsymbol{\alpha}} m_1(\mathbf{n}) = \frac{P_{\boldsymbol{\alpha}}(\mathbf{n})}{|\mathbf{n}|^{2\alpha+2\beta+1}},$$

where $P_{\boldsymbol{\alpha}}$ is a homogeneous polynomial of degree $\alpha+\beta+1$, and

$$\left(\frac{\partial}{\partial x}\right)^{\boldsymbol{\alpha}} m_{12}(\mathbf{n}) = \frac{Q_{\boldsymbol{\alpha}}(\mathbf{n})}{|\mathbf{n}|^{2(\alpha+\beta+1)}},$$

where $Q_{\boldsymbol{\alpha}}$ is a homogeneous polynomial of degree $\alpha+\beta+2$.

From these relations it easily follows that there are constants $B_{\boldsymbol{\alpha}}$ and $C_{\boldsymbol{\alpha}}$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^{\boldsymbol{\alpha}} m_1(\mathbf{n})\right| \leqslant \frac{B_{\boldsymbol{\alpha}}}{|x|^{\alpha}|y|^{\beta}}$$

and

$$\left|\left(\frac{\partial}{\partial x}\right)^{\boldsymbol{\alpha}} m_{12}(\mathbf{n})\right| \leqslant \frac{C_{\boldsymbol{\alpha}}}{|x|^{\alpha}|y|^{\beta}}.$$

In particular, these inequalities are true for $0\leqslant\alpha\leqslant 14$, $0\leqslant\beta\leqslant 14$. It then can be shown by transferring lemma 8 of [5] from $\mathbb{R}^2$ to T^2 that at every point $\mathbf{x}$ of T^2

$$A_{\mathbf{0}}(u_1)(\mathbf{x}) < c g^*_{28}(u)(\mathbf{x})$$

and

$$A_{\mathbf{0}}(u_{12})(\mathbf{x}) < c g^*_{28}(u)(\mathbf{x}).$$

Theorem 4 on page 1029 of [5] states that $N_{\mathbf{0}}(v)$ is finite almost everywhere that $A_{\mathbf{0}}(v)$ is, for any harmonic function v. Thus lemma B follows from the last two inequalities.

Both lemma B and lemma 8 of [5] are proved following the reasoning used in the proof of relation (41) of section 3.4.1 of [12], p. 233-235. That reasoning used lemma 2 of section 2.5.2 of [12], p. 216-217. The proof of lemma 2 given there is in slight error. Professor Stein has orally communicated a corrected version to us. To ease our conscience over failing to present a detailed proof of lemma B, we will prove lemma 2′, which is a correct version of Stein's lemma 2. Our notation for lemma 2′ will be local to its proof only.

LEMMA 2′ [13]. — *Let* $\mathbf{x}\in\mathbb{R}^n$, $y>0$, *and* $\Gamma_\beta=\{(\mathbf{x},y)\in\mathbb{R}_+^{n+1}\,|\,|\mathbf{x}|<\beta y\}$ *be a cone with vertex* $\mathbf{0}$. *With* $x_0=y$, *write*

$$|\nabla^k u|^2=\sum_{j=0}^n\left|\nabla^{k-1}\left(\frac{\partial u}{\partial x_j}\right)\right|^2,\qquad k=1,2,3,\ldots$$

Suppose u is harmonic in Γ_β *and* $|\nabla u|\to 0$ *as* $y\to\infty$, *for* $(\mathbf{x},y)\in\Gamma_\beta$. *Then for each* $k\geqslant 1$,

$$(27)\qquad \iint_{\Gamma_\beta}|\nabla u|^2 y^{1-n}\,d\mathbf{x}\,dy\leqslant c_k\iint_{\Gamma_\beta}|\nabla^k u|^2 y^{2k-n-1}\,d\mathbf{x}\,dy,$$

where $c_k=\prod_{m=1}^{k-1}(2m-1)^{-1}$.

Proof. — It suffices to prove that for $j=0,1,\ldots,n$ and for $m=1,2,\ldots,k-1$ (assume $k\geqslant 2$, since $k=1$ is trivial)

$$(28_{m,j})\qquad \iint_{\Gamma_\beta}\left|\frac{\partial^m u}{\partial x_j\,\partial y^{m-1}}\right|^2 y^{2m-1-n}\,d\mathbf{x}\,dy$$

$$\leqslant\frac{1}{(2m-1)}\iint_{\Gamma_\beta}\left|\frac{\partial^{m+1}u}{\partial x_j\,\partial y^m}\right|^2 y^{2m+1-n}\,d\mathbf{x}\,dy.$$

For fixing j and concatenating equations $(28_{m,j})$ for $m=1,2,\ldots,k-1$, we get

$$(29_j)\qquad \iint_{\Gamma_\beta}\left|\frac{\partial u}{\partial x_j}\right|^2 y^{1-n}\,dy\leqslant c_k\int_{\Gamma_\beta}\iint\left|\frac{\partial^k u}{\partial y^{k-1}\,\partial x_j}\right|^2 y^{2k-1-n}\,dy,$$

for $j=0,1,\ldots,n$. Sum inequalities (29_j) for $j=0,1,\ldots,n$.

Inequality (27) follows since

$$\sum_{j=0}^n\left|\frac{\partial u}{\partial x_j}\right|^2=|\nabla u|^2,\qquad\text{while}\qquad \sum_{j=0}^n\left|\frac{\partial^k u}{\partial y^{k-1}\,\partial x_j}\right|^2\leqslant|\nabla^k u|^2.$$

The hyperplane $y=1$ intersects Γ_β in the n-disk $D:=\{(\mathbf{v},1):\sqrt{v_1^2+\ldots+v_n^2}<\beta\}$ so that Γ_β is a union of rays of the form

$\{(y\mathbf{v},y)\,|\,y>0\}$ with each ray starting at 0 and passing through D at $(\mathbf{v},1)$. To verify equations (28) fix m and j and let $f:=\partial^m u/\partial x_j\,\partial y^{m-1}$. For each $\mathbf{v}$, $|\mathbf{v}|<\beta$, the distance from $(y\mathbf{v},y)$ to the complement of Γ_β becomes larger than 1 if y is sufficiently large. Hence if B_y is a ball of radius 1 about $(y\mathbf{v},y)$, our hypothesis implies $\sup_{\mathbf{z}\in B_y}|\nabla u(\mathbf{z})|\to 0$ as $y\to\infty$. This in turn implies that $\lim_{y\to\infty}\partial f/\partial y\,(y\mathbf{v},y)=0$. (*See* [12], p. 216, formula (18).) Hence

$$f(y\mathbf{v},y)=-\int_y^\infty\frac{\partial f}{\partial s}(s\mathbf{v},s)\,s^m s^{-m}\,ds.$$

By Schwarz's inequality, therefore,

$$|f(y\mathbf{v},y)|^2\leqslant\int_y^\infty\left|\frac{\partial f}{\partial s}(s\mathbf{v},s)\right|^2 s^{2m}\,ds\int_y^\infty s^{-2m}\,ds.$$

Multiply by y^{2m-1} and integrate from 0 to ∞ to get

$$\int_0^\infty|f(y\mathbf{v},y)|^2 y^{2m-1}\,dy$$

$$\leqslant\frac{1}{2m-1}\int_0^\infty\left(\int_y^\infty\left|\frac{\partial f}{\partial s}(s\mathbf{v},s)\right|^2 s^{2m}\,ds\right)dy$$

$$=\frac{1}{2m-1}\int_0^\infty\left(\int_0^\infty\left|\frac{\partial f}{\partial s}(s\mathbf{v},s)\right|^2 s^{2m}\chi_{\{s>y\}}\,ds\right)dy$$

$$=\frac{1}{2m-1}\int_0^\infty\left|\frac{\partial f}{\partial y}(y\mathbf{v},y)\right|^2 y^{2m+1}\,dy.$$

The last equality is immediate from interchanging the order of integration. Finally integrate over $\{|\mathbf{v}|<\beta\}$ and make the change of variables $\mathbf{x}=y\mathbf{v}$, $y^{-n}\,d\mathbf{x}\,dy=d\mathbf{v}\,dy$ to obtain equations $(28_{m,j})$. Lemma 2′ is proved.

Proof of Lemma C. — The proof of this Tauberian lemma involves several applications of the summation by parts formula in two variables, that is to say, equation (0).

We also need the following simple lemma:

LEMMA D. — *If*

$$f(x,y)=\frac{x}{m}+\frac{y}{n}+(1-1/m)^x(1-1/n)^y-1$$

then for x *and* y *non-negative* $x\notin(0,1)$, $y\notin(0,1)$ *and* $m\geqslant 1$, $n\geqslant 1$ *we have* $f(x,y)\geqslant 0$.

Proof of Lemma D. — Note that $f(0,0)=f(1,0)=0$. We also have

$$\frac{\partial^2}{\partial x^2}f(x,y)=(1-1/m)^x(1-1/n)^y\ln^2(1-1/m)\geqslant 0.$$

Consequently $f(.,0)$ is convex. Hence $f(x,y)\geqslant 0$ on $[1,\infty)\times\{0\}$.

Similarly $f(x,y)\geqslant 0$ on $\{0\}\times[1,\infty)$. For x and y in $[1,\infty)\times[1,\infty)$ we can write

$$f(x,y)\geqslant f(x,y)-f(x,0)-f(0,y)+f(0,0).$$

However

$$f(x,y)-f(x,0)-f(0,y)+f(0,0)=xy\frac{\partial^2 f}{\partial x\,\partial y}(\varphi x,\eta y),$$

for some choice of φ and η between 0 and 1, and

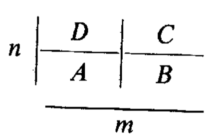
$$\frac{\partial^2 f}{\partial x\,\partial y}(\varphi x,\eta y)=(1-1/m)^{\varphi x}(1-1/n)^{\eta y}\ln(1-1/m)\ln(1-1/n)\geqslant 0.$$

We now continue with the proof of Lemma C. We will show that

$$\sum_{j=1}^{m}\sum_{k=1}^{n}a_{jk}-\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}r^j s^k\to 0, \tag{30}$$

with $r=1-1/m$ and $s=1-1/n$. Given $\varepsilon>0$, select p so large that if $m\geqslant p^5$ and $n\geqslant p^5$ then $|\varepsilon_{mn}|<\varepsilon$. We can also assume that $1/p<\varepsilon$.

There are four regions that must be studied. We will assume that $m>p^5$, $n>p^5$.



Region A is the only place where both of the sums of expression (30) are involved. We first study

$$\sum_{j=1}^{m}\sum_{k=1}^{n}a_{jk}(1-r^j s^k) \tag{31}$$

$$=\sum_{j=1}^{m}\sum_{k=1}^{n}\left\{\frac{1-r^j s^k}{j^2+k^2}\right\}\{(j^2+k^2)\,a_{jk}\}$$

$$=\sum_{j=1}^{m}\sum_{k=1}^{n}\lambda_{jk}c_{jk},$$

where

$$\lambda_{jk}=\frac{1-r^j s^k}{j^2+k^2}\qquad\text{and}\qquad c_{jk}=(j^2+k^2)\,a_{jk}.$$

We now use the summation by parts formula (0) and the estimates

$$|\Delta_{jk}^{11}|<C\left(\frac{r^j s^k}{mn(j^2+k^2)}+\frac{kr^j s^k}{m(j^2+k^2)^2}+\frac{jr^j s^k}{n(j^2+k^2)^2}+\frac{jk(1-r^j s^k)}{(j^2+k^2)^3}\right)$$

$$|\Delta_{jk}^{10}|<C\left(\frac{j(1-r^j s^k)}{(j^2+k^2)^2}+\frac{r^j s^k}{m(j^2+k^2)}\right)$$

and

$$|\Delta_{jk}^{01}|<C\left(\frac{k(1-r^j s^k)}{(j^2+k^2)^2}+\frac{r^j s^k}{n(j^2+k^2)}\right).$$

The first term of the summation by parts formula in region A is

$$\left|\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\Delta_{jk}^{11}s_{11}^{jk}\right|\leqslant C\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\frac{r^j s^k}{mn(j^2+k^2)}|\varepsilon_{jk}|jk$$

$$+C\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\frac{kr^j s^k}{m(j^2+k^2)^2}|\varepsilon_{jk}|jk$$

$$+C\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\frac{jr^j s^k}{n(j^2+k^2)^2}|\varepsilon_{jk}|jk$$

$$+C\sum_{j=1}^{m-1}\sum_{k=1}^{n-1}\frac{jk(1-r^j s^k)}{(j^2+k^2)^3}|\varepsilon_{jk}|jk$$

$$=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

We now estimate each piece. First,

$$I \leqslant C \sum_{j=1}^{m} \sum_{k=1}^{p} \frac{r^j s^k jk}{mn(j^2+k^2)}$$

$$+ C \sum_{j=1}^{p} \sum_{k=1}^{n} \frac{r^j s^k jk}{mn(j^2+k^2)}$$

$$+ \varepsilon C \sum_{j=n}^{m} \sum_{k=p}^{n} \frac{r^j s^k jk}{mn(j^2+k^2)}$$

$$= I_1 + I_2 + I_3.$$

It is necessary to consider each of these pieces because the estimate $|\varepsilon_{jk}| < \varepsilon jk$ is not valid if j or k is too small. For small j, k the best we know is that $|\varepsilon_{jk}| < C jk$.

Since

$$\left(\frac{jk}{j^2+k^2}\right) r^j s^k < 1,\; I_1 < C \frac{1}{mn} mp < C \frac{p}{m} < \frac{C}{p^4} < C\varepsilon.$$

The second term, I_2 is estimated in exactly the same way. We also have

$$I_3 < \varepsilon C \frac{1}{mn} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} r^j s^k = C\varepsilon.$$

(Recall the choice of r and s.)

To estimate II, we again break up the sum into three pieces:

$$\mathrm{II} \leqslant C \sum_{j=1}^{m} \sum_{k=1}^{p} \frac{jk^2 r^j s^k}{m(j^2+k^2)^2}$$

$$+ C \sum_{j=1}^{p} \sum_{k=1}^{n} \frac{jk^2 r^j s^k}{m(j^2+k^2)^2}$$

$$+ \varepsilon C \sum_{j=p}^{m} \sum_{k=p}^{n} \frac{jk^2 r^j s^k}{m(j^2+k^2)^2}$$

$$= \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3.$$

The term II_1 can be estimated by

$$\mathrm{II}_1 \leqslant C \sum_{k=1}^{p} \frac{1}{m} \int_0^{\infty} \frac{jk^2}{(j^2+k^2)^2} dj \leqslant C \sum_{k=1}^{p} \frac{1}{m} \int_0^{\infty} \frac{y}{(1+y^2)^2} dy \leqslant C \frac{p}{m} < C\varepsilon,$$

where the substitutions $j = ky$ and $dj = k dy$ stet used. Using the same method, we can bound both II_2 and II_3 by $C\varepsilon$.

The estimate for III is exactly the same as that for II with the rolls of j and k reversed.

The last term in our expression, IV, is also broken up into three pieces. Since IV involves the expression $(1 - r^j s^k)$, with $r = 1 - 1/m$ and $s = 1 - 1/n$, Lemma D implies that we can replace $1 - r^j s^k$ with $(j/m + k/n)$. Thus,

$$\mathrm{IV} \leqslant C \sum_{j=1}^{m} \sum_{k=1}^{p} \frac{j^2 k^2 j/m}{(j^2+k^2)^3}$$

$$+ C \sum_{j=1}^{p} \sum_{k=1}^{n} \frac{j^2 k^2 j/m}{(j^2+k^2)^3} + \varepsilon C \sum_{j=p}^{m} \sum_{k=p}^{n} \frac{j^2 k^2 j/m}{(j^2+k^2)^3}$$

+ similar terms with j/m replaced by k/n.

The first of these terms is estimated by

$$C \sum_{k=1}^{p} \frac{1}{m} \int_0^{\infty} \frac{j^3 k^2}{(j^2+k^2)^3} dj$$

which by a change of variables, is seen to be controlled by $Cp/m < C\varepsilon$.

The second term can be estimated by

$$C \sum_{k=1}^{n} \frac{1}{m} \int_0^{p} \frac{j^3 k^2}{(j^2+k^2)^3} dj.$$

Let $j = ky$, $dj = k dy$, and we can bound this second term by

$$\frac{C}{m} \sum_{k=1}^{n} \int_0^{p/k} \frac{y^3 dy}{(1+y^2)^3} < \frac{C}{m} \sum_{k=1}^{n} (p/k)^4 < Cp^4/m < C\varepsilon.$$

The third piece is bounded by

$$\frac{C\varepsilon}{m} \int_p^m \int_p^n \frac{j^3 k^2}{(j^2+k^2)^3} dk\, dj.$$

If $jy=k$, $jdy=dk$, this is less than

$$\frac{C\varepsilon}{m}\int_p^m\int_0^\infty \frac{y^3}{(y^2+1)^3}\,dy\,dj<\frac{C\varepsilon}{m}\int_p^m d_j\leqslant\frac{m\,C\varepsilon}{m}=C\varepsilon.$$

This and the fact that the terms with j/m replaced by k/n yield the same estimates concludes the estimate for IV.

The second term in the summation by parts formula for region A is estimated by

$$\sum_{j=1}^{m-1}\Delta_{jn}^{10}\,s_{11}^{jn}<C\sum_{j=1}^{m}\left\{\frac{j(1-r^j s^n)}{(n^2+j^2)^2}+\frac{r^j s^n}{m(n^2+j^2)}\right\}jn\,\varepsilon_{jn}.$$

This by Lemma D, and the definition of p, is less than

$$C\left[\sum_{j=1}^{p}\frac{j^2 n((j/m)+1)}{(n^2+j^2)^2}+\varepsilon\sum_{j=p}^{m}\frac{j^2 n((j/m)+1)}{(n^2+j^2)^2}\right.$$
$$\left.+\sum_{j=1}^{p}\frac{jnr^j s^k}{m(n^2+j^2)}+\varepsilon\sum_{j=p}^{m}\frac{jnr^j s^k}{m(n^2+j^2)}\right].$$

Estimate the first two sums by the corresponding integrals, using the substitutions $j=ny$, and $dj=ndy$. Bound $jn/(n^2+j^2)$ by 1 in the last two sums. The upper bound for this term is now

$$C\left[\int_0^{p/n}\frac{n}{m}\frac{y^3}{(1+y^2)^2}\,dy+\int_0^{p/n}\frac{y^2}{(1+y^2)^2}\,dy\right.$$
$$+\varepsilon\int_{p/n}^{m/n}\frac{n}{m}\frac{y^3}{(1+y^2)^2}\,dy+\varepsilon\int_{p/n}^{m/n}\frac{y^2}{(1+y^2)^2}\,dy$$
$$\left.+\sum_{j=1}^{p}\frac{1}{m}+\frac{\varepsilon}{m}\sum_{j=p}^{m}r^j\right].$$

For the first and third integrals use $y^3(1+y^2)^{-2}<1$. For the second use $y^2(1+y^2)^{-2}<y^2$. In the last integral expand the interval of integration to $(0,\infty)$. In the last sum expand the limits of summation to 0 and ∞. This term is then bounded by

$$C\left[\frac{n}{m}\frac{p}{n}+\frac{1}{3}\left(\frac{p}{n}\right)^3+\varepsilon\frac{n}{m}\frac{m}{n}+C\varepsilon+\frac{p}{m}+\frac{\varepsilon}{m}m\right]<C\varepsilon.$$

The third term in the summation by parts formula for region A is exactly symmetrical to the second. Simply interchange j and k, and m and n. The estimate is then exactly as above.

The last term in the summation by parts formula for region A is

$$\lambda_{mn}s_{11}^{mn}=\frac{1-r^m s^n}{m^2+n^2}mn\,\varepsilon_{mn}<\frac{mn}{m^2+n^2}\varepsilon<\varepsilon.$$

This completes the proof for region A.

For region B we must study

$$\sum_{j=m}^{\infty}\sum_{k=1}^{n}\left[\frac{r^j s^k}{j^2+k^2}\right](j^2+k^2)\,a_{jk}$$
$$=\sum_{j=m}^{\infty}\sum_{k=1}^{n}\lambda_{jk}\,c_{jk}\qquad\text{where}\quad \lambda_{jk}=\frac{r^j s^k}{j^2+k^2}.$$

Again apply (0). We need estimates for Δ_{jk}^{11}, Δ_{jk}^{10}, and Δ_{jk}^{01}.

These are:

$$|\Delta_{jk}^{11}|<Cr^j s^k\left[\frac{1}{mn(j^2+k^2)}+\frac{k}{m(j^2+k^2)^2}\right.$$
$$\left.+\frac{j}{n(j^2+k^2)^2}+\frac{1}{(j^2+k^2)^2}\right],$$

$$|\Delta_{jk}^{10}|<Cr^j s^k\left[\frac{1}{(j^2+k^2)^2}+\frac{1}{m(j^2+k^2)}\right]$$

and

$$|\Delta_{jk}^{01}|<Cr^j s^k\left[\frac{k}{(j^2+k^2)^2}+\frac{1}{n(j^2+k^2)}\right].$$

The first term in the summation by parts formula for region B is estimated by

$$C\sum_{j=m}^{\infty}\sum_{k=1}^{n-1}\frac{r^j s^k\,|s_{m\,1}^{jk}|}{mn(j^2+k^2)^2}+C\sum_{j=m}^{\infty}\sum_{k=1}^{n-1}\frac{r^j s^k\,|s_{m\,1}^{jk}|\,k}{m(j^2+k^2)^2}$$
$$+C\sum_{j=m}^{\infty}\sum_{k=1}^{n-1}\frac{r^j s^k\,|s_{m\,1}^{jk}|\,j}{n(j^2+k^2)^2}$$

$$+C\sum_{j=m}^{\infty}\sum_{k=1}^{n-1}\frac{r^j s^k |s_{m\,1}^{jk}|}{(j^2+k^2)^2}=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.$$

Since $s_{m\,1}^{jk}=s_{11}^{jk}-s_{11}^{(m-1)\,k}$, and $m-1$ and j exceed p, we may bound $|s_{m\,1}^{jk}|$ by Cjk if $k\leqslant p$ and by $C\varepsilon jk$ if $k>p$.

To estimate I we have:

$$\mathrm{I}<C\sum_{j=m}^{\infty}\sum_{k=1}^{p}\frac{jk\,r^j s^k}{mn(j^2+k^2)}$$

$$+C\varepsilon\sum_{j=m}^{\infty}\sum_{k=p}^{n-1}\frac{jk\,r^j s^k}{mn(j^2+k^2)}<\frac{Cp}{mn}\sum_{j=0}^{\infty}r^j$$

$$+\frac{C\varepsilon}{mn}\sum_{j=0}^{\infty}r^j\sum_{k=0}^{\infty}s^k<\frac{Cp}{mn}m+\frac{C\varepsilon}{mn}mn<C\varepsilon.$$

To estimate II we have

$$\mathrm{II}<C\sum_{j=m}^{\infty}\sum_{k=1}^{p}\frac{jk^2 r^j s^k}{m(j^2+k^2)^2}$$

$$+C\varepsilon\sum_{j=m}^{\infty}\sum_{k=p}^{n}\frac{jk^2 r^j s^k}{m(j^2+k^2)^2}<C\sum_{j=m}^{\infty}\frac{1}{m}\int_0^p\frac{jk^2}{(j^2+k^2)^2}dk$$

$$+\frac{C\varepsilon}{m}\sum_{j=m}^{\infty}r^j\int_p^n\frac{jk^2}{(j^2+k^2)^2}dk.$$

Using the substitutions $k=jy$ and $dk=j\,dy$ yields

$$\mathrm{II}<\frac{C}{m}\sum_{j=m}^{\infty}\int_0^{p/j}\frac{y^2\,dy}{(1+y^2)^2}+\frac{C\varepsilon}{m}\sum_{j=m}^{\infty}r^j\int_{p/j}^{n/j}\frac{y^2\,dy}{(1+y^2)^2}.$$

Replace the denominator in the first integral by 1, and extend the limits of the second integral to $(0,\infty)$ to obtain

$$\mathrm{II}<\frac{C}{m}\sum_{j=m}^{\infty}\frac{p^3}{j^3}+\frac{C\varepsilon}{m}\sum_{j=0}^{\infty}r^j<\frac{Cp^3}{m\cdot m^2}+C\varepsilon<C\varepsilon.$$

For III we have

$$\mathrm{III}<C\sum_{j=m}^{\infty}\sum_{k=1}^{p}\frac{j^2 k r^j s^k}{n(j^2+k^2)^2}$$

$$+C\varepsilon\sum_{j=m}^{\infty}\sum_{k=p}^{n}\frac{j^2 k r^j s^k}{n(j^2+k^2)^2}$$

$$<C\sum_{k=1}^{p}\frac{1}{n}\int_m^{\infty}\frac{j^2 k\,dj}{(j^2+k^2)^2}$$

$$+C\frac{\varepsilon}{n}\sum_{k=p}^{n}s^k\int_m^{\infty}\frac{j^2 k\,dj}{(j^2+k^2)^2}.$$

Let $j=ky$ $dj=k\,dy$ and extend both domains of integration to $[0,\infty)$, and extend the second sum from 0 to ∞, to get

$$\mathrm{III}<C\sum_{k=1}^{p}\frac{1}{n}\int_0^{\infty}\frac{y^2\,dy}{(1+y^2)^2}+C\frac{\varepsilon}{n}\sum_{k=0}^{\infty}s^k\int_0^{\infty}\frac{y^2\,dy}{(1+y^2)^2}<C\frac{p}{n}+C\varepsilon<C\varepsilon.$$

Upon bounding $jk/(j^2+k^2)$ by C, we have

$$\mathrm{IV}<C\sum_{j=m}^{\infty}\int_0^p\frac{dk}{j^2+k^2}+C\varepsilon\sum_{j=m}^{\infty}r^j\int_p^n\frac{dk}{j^2+k^2}.$$

Expand the limits of the second integral to $[0,\infty)$, and use the substitutions $k=yj$ and $dk=j\,dy$ in both integrals to get

$$\mathrm{IV}<C\sum_{j=m}^{\infty}\frac{1}{j}\int_0^{p/j}\frac{dy}{1+y^2}+C\varepsilon\sum_{j=m}^{\infty}\frac{r^j}{j}\int_0^{\infty}\frac{dy}{1+y^2}$$

$$<C\sum_{j=m}^{\infty}\frac{p}{j^2}+C\frac{\varepsilon}{m}\sum_{j=0}^{\infty}r^j<C\frac{p}{m}+C\varepsilon<C\varepsilon.$$

Now consider the second term in the summation by parts formula for region B. This is less than

$$C\varepsilon\sum_{j=m}^{\infty}\frac{j^2\,n\,r^j s^k}{(j^2+n^2)^2}+C\varepsilon\sum_{j=m}^{\infty}\frac{jnr^j s^k}{m(j^2+n^2)}$$

$$<C\varepsilon\int_0^{\infty}\frac{j^2\,n\,dj}{(j^2+n^2)^2}+C\varepsilon\frac{1}{m}\sum_{j=m}^{\infty}r^j.$$

Letting $j=ny$ in the integral, and using the fact that in the region of interest, $1/j<1/m$ allows us to estimate this term by

$$C\varepsilon\int_0^{\infty}\frac{y^2\,dy}{1+y^2}+\frac{C\varepsilon}{m}\sum_{j=0}^{\infty}r^j<C\varepsilon.$$

The third and fourth terms in the summation by parts formula are zero since for fixed k,

$$\lim_{M\to\infty} r^M s^k \left[\frac{k}{(M^2+k^2)^2}+\frac{1}{n(M^2+k^2)}\right]=0,$$

and

$$\lim_{M\to\infty} r^m s^k \left[\frac{1}{M^2+k^2}\right]=0.$$

For region C write

$$\sum_{j=m}^{\infty} \sum_{k=n+1}^{N} c_{jk}\lambda_{jk}=\sum_{j=m}^{\infty} \sum_{k=0}^{N}-\sum_{j=m}^{\infty} \sum_{k=0}^{n}=\sum_N-\sum_n.$$

By the argument for region B, $|\sum_N|<C\varepsilon$ and $|\sum_n|<C\varepsilon$ where C is independent of N. Letting N go to infinity shows the whole sum to be controlled by $C\varepsilon$ also.

Region D is the same as region B with the two coordinates interchanged. This completes the proof of Lemma C.

Proof of theorem 1. — Let u be the Poisson integral of s. By theorem 2, $g_{28}^*(u)<\infty$ a. e. on E. By lemma B, $N_0(u_{12})<\infty$ a. e. on E. By Théorème [4], p. 95, u_{12} converges a. e. on E non-tangentially and a fortiori radially, that is to say rectangularly unrestrictedly Abel. In other words, there is a set $F\subset E$ so that the series $s_{12}(\mathbf{x})=-\Sigma\, mn/|\mathbf{n}|^2 D_{\mathbf{n}}(\mathbf{x})$ is rectangularly unrestrictedly Abel summable for all $\mathbf{x}\in F$ and $|E|=|F|$. By lemma A, parts (6) and (8) we also know that there is a set $G\subset E$, $|E|=|G|$, so that $1/MN\sum_{\mathbf{0}\leq\mathbf{n}\leq\mathbf{N}} mn\, D_{\mathbf{n}}(\mathbf{x})$ is bounded and converges to 0 as $\|\mathbf{N}\|\to\infty$ at each point of G. (The bound and the rate of convergence may vary as $\mathbf{x}$ varies over G.) By lemma C, $s_{12}(\mathbf{x})$ converges for every $\mathbf{x}$ in $F\cap G$. Since $|F\cap G|=|E|$, theorem 1 is proved.

Example. — One might expect that an argument similar to that given above would work for the discrete analogues of the two single Riesz transforms, namely the multipliers

$$\frac{m}{\sqrt{m^2+n^2}} \quad\text{and}\quad \frac{n}{\sqrt{m^2+n^2}}.$$

For these everything works as above until the Tauberian theorem C. The natural Tauberian condition that arises for the first multiplier is

$$\frac{1}{M}\sum_{j=1}^{M} \sum_{k=1}^{N} \sqrt{j^2+k^2}\, a_{jk}\to 0$$

and the natural Tauberian condition that arises for the second is

$$\frac{1}{N}\sum_{j=1}^{M} \sum_{k=1}^{N} \sqrt{j^2+k^2}\, a_{jk}\to 0.$$

Unfortunately, these Tauberian conditions are too weak. The following example shows that the Tauberian condition associated with the second multiplier can hold for an unrestrictedly Abel summable but UR divergent series. (An example for the first multiplier can be obtained by interchanging the roles of M and N in the example given.) The series fails to converge UR by having a column of positive terms followed immediately by a similar column of negative terms. Selecting M so that only one column from such a pair is included leads to a large partial sum. Increasing M by 1 to include the additional column leads to a partial sum of zero. The series is Abel summable to zero because we use both columns of every column pair, and for r close to 1, r^k and r^{k+1} are almost the same. The Tauberian condition is satisfied because if we use both columns of a column pair, the sum is small, and if we split a column pair, the partial sum is not too large, and dividing by N forces it to zero. The details are given below.

There is a numerical series $S=\sum a_{jk}$ satisfying

(32) S is rectangularly unrestrictedly Abel summable to 0.

(33) $\frac{1}{N}\sum_{j=1}^{M} \sum_{k=1}^{N} \sqrt{j^2+k^2}\, a_{jk}\to 0$ as $\|M, N\|=\min\{M, N\}\to\infty$.

(34) S diverges UR.

Define $S=\sum a_{jk}$ as follows:

$$a_{2^j-1,\, 2^k}=\frac{1}{k} \quad\text{for}\quad j\in\{1, 2, 3, \ldots\}$$

and

$$k\in\{j, j+1, \ldots, 2j\}$$

$$a_{2^j,\,2^k} = \frac{-1}{k} \qquad \text{for} \quad j \in \{1, 2, 3, \ldots\}$$

and

$$k \in \{j, j+1, \ldots, 2j\}$$

$$a_{j,\,k} = 0 \text{ otherwise.}$$

Proof. — For (34) note that

$$\sum_{j=1}^{2m} \sum_{k=1}^{2^{2m}} a_{j,\,k} - \sum_{j=1}^{2m} \sum_{k=1}^{2^{2m-1}} a_{j,\,k} = 0 - \sum_{j=m}^{2m} \frac{1}{j} \to -\ln(2).$$

For (32) we must estimate

$$(35) \quad \sum_{j=1}^{\infty} \sum_{k=j}^{2j} \frac{1}{k} r^{2^j-1} s^{2^k} - \frac{1}{k} r^{2^j} s^{2^k} |$$

$$= \frac{1-r}{r} \sum_{j=1}^{\infty} \sum_{k=j}^{2j} \frac{1}{k} r^{2^j} s^{2^k} < C(1-r) \sum_{j=1}^{\infty} r^{2^j}$$

$$< C(1-r) [\sum_{j=1}^{2^N} r^j + \sum_{j=N+1}^{\infty} r^{2^j}]$$

$$< C(1-r) \left[\frac{1-r^{2^N}}{1-r} r + r^{2^N} \sum_{j=0}^{\infty} \{ r^{2^N} \}^j \right]$$

$$< C(1-r^{2^N}) + \frac{C(1-r) r^{2^N}}{1-r^{2^N}}.$$

But,

$$1-r^{2^N} = (1-r)(1+r+r^2+\ldots+r^{2^N-1})$$

$$>(1-r) r^{2^N} (1+\ldots+1)$$

$$\geqq (1-r) 2^N r^{2^N}.$$

Using this we obtain an estimate for (35) of $C(1-r^{2^N}) + (C/2^N)$.

Given $\varepsilon > 0$, pick N so that $C/2^N < \varepsilon/2$. Then pick r so close to 1 that $C(1-r^{2^N}) < \varepsilon/2$.

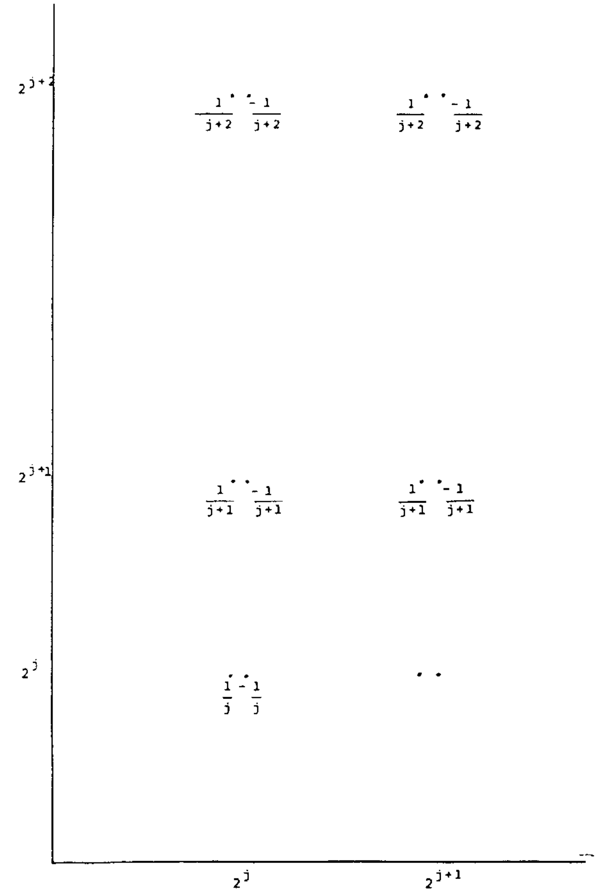


Fig. 1.

For (33) we must show that

$$\frac{1}{N} \sum_{j=1}^{M} \sum_{k=1}^{N} \sqrt{j^2+k^2}\, a_{jk} - \varepsilon_{MN} \to 0$$

U.R. If $M > N$ then $N \varepsilon_{MN} = N \varepsilon_{NN}$ (See *Fig.* 1), whence $\varepsilon_{MN} = \varepsilon_{NN}$ so it suffices to show that $\varepsilon_{MN} \to 0$ U.R. for $N \geqq M$. Choose m a positive integer so that $2^{m-1} \leqq M < 2^m$, and define n (not necessarily an integer) so that $2^n = N$. Define

$$\delta = \begin{cases} 0 & \text{if } M \neq 2^m - 1 \\ 1 & \text{otherwise} \end{cases}. \quad \text{Then}$$

$$\varepsilon_{MN} = \frac{1}{2^n}\Bigg[\sum_{j=1}^{m-1} \sum_{k=j}^{\|2j,\,[n]\|}\left(\frac{1}{k}\sqrt{(2^j-1)^2+(2^k)^2} - \frac{1}{k}\sqrt{(2^j)^2+(2^k)^2}\right) + \delta \sum_{k=m}^{\|2m,\,[n]\|}\frac{1}{k}\sqrt{(2^n-1)^2+(2^k)^2}\Bigg],$$

where $\|2j, [n]\| = \min\{2j, [n]\}$. Use

$$\left|\sqrt{(x-1)^2-a^2}-\sqrt{x^2-a^2}\right| \langle 1 \text{ for } x \rangle 1 \text{ and } \|2m, [n]\| \leqq 2m$$

for the first term, and $|\delta| \leqq 1$ and $\|2m, [n]\| \leqq [n]$ for the second to get

$$|\varepsilon_{MN}| < \frac{1}{2^n}\left[\sum_{j=1}^{m-1}\sum_{k=j}^{2j}\frac{1}{k} + \sum_{k=m}^{[n]}\frac{1}{m}\sqrt{2.2^{2k}}\right]$$

$$< \frac{C}{2^n}\left[(m-1)\ln(2) + \frac{\sqrt{2}}{m}\sum_{k=m}^{[n]} 2^k\right]$$

$$< \frac{C}{2^n}\left[n\ln(2) + \frac{\sqrt{2}\,2^n}{\mathrm{m}}\sum_{k=0}^{\infty} 2^{-k}\right]$$

$$< C\left[\frac{n}{2^n} + \frac{1}{m}\right],$$

which goes to zero as $\|M, N\|$ goes to ∞.

Appendix

GEOMETRIC LEMMA 1. — $\Delta_m \subseteq \Gamma_m \subseteq \Delta_{2m+4}$.

Proof. — For $0 \leqq \alpha \leqq \pi/2$, we have

(36) $$1 - \frac{\alpha^2}{2} \leqq \cos\alpha \leqq 1 - \frac{\alpha^2}{\pi},$$

since the functions $1-(\alpha^2/\pi)-\cos\alpha$ and $\cos\alpha+(\alpha^2/2)-1$ are both 0 at $\alpha=0$ and have positive derivatives on $(0, \pi/2)$. Let $z = re^{i\theta} \in \Delta_m$. We may assume $0 \leqq \theta \leqq \pi$. If $\alpha \leqq \theta \leqq \pi$ (recall $\alpha = \pi/4 \cdot 1/2^m$), then

$$r < 1 - \theta\, 2^{-m} \leqq 1 - \alpha\, 2^{-m} = 1 - \frac{4}{\pi}\alpha^2.$$

Hence by (36), $r < \cos\alpha$ and $z \in \Gamma_m$.

Now look at the functions

$$g(\alpha, \theta) := -3\alpha\theta^2 + (4\alpha^2+1)\theta + \alpha(1-\alpha^2)$$

and

$$f(\alpha, \theta) := \cos\alpha - \cos(\alpha-\theta)(1-\theta\, 2^{-m})$$

on the triangular region

$$T = \left\{(\alpha, \theta) \,\middle|\, 0 \leqq \theta \leqq \alpha \leqq \frac{\pi}{4}\right\}$$

in the (α, θ) plane. Since $g = 2\alpha \geqq 0$ on the top of

$T(=\{(\alpha, \alpha)\})$, $\quad g = \alpha(1-\alpha^2) \geqq 0$

on the bottom of

$$T\left(=\left\{(\alpha, 0) \,\middle|\, 0 \leqq \alpha \leqq \frac{\pi}{4}\right\}\right),$$

and

$g_{\theta\theta} = -6\alpha \leqq 0$ on T; $\quad g \geqq 0$ on T.

Also

$$f_\theta = -\sin(\alpha-\theta)(1-2\alpha\theta) + 2\alpha\cos(\alpha-\theta)$$

so that

$f_\theta \geqq 0$ on $T \cap \{1 - 2\alpha\theta \leqq 0\}$,

whereas if

$(\alpha, \theta) \in T \cap \{1-2\alpha\theta>0\}$ from (36),

$$f_\theta \geqq -(\alpha-\theta)(1-2\alpha\theta)+2\alpha\left[1-\frac{(\alpha-\theta)^2}{2}\right]=g(\alpha, \theta)\geqq 0.$$

Since $f=0$ on the bottom of T and $f_\theta \geqq 0$ on T, $f \geqq 0$ on all of T so that

$$1-\theta 2^{-m} \leqq \frac{\cos\alpha}{\cos(\alpha-\theta)} \quad \text{which implies } \Delta_m \subseteq \Gamma_m.$$

Now let $z=re^{i\theta}\in\Gamma_m$. Assume $0\leqq\theta\leqq\pi$. If $\alpha\leqq\theta\leqq\pi$, $r\leqq\cos\alpha$. By (36), $\cos\alpha\leqq 1-(\alpha^2/\pi^2)\pi$. From $\alpha^2/\pi^2=2^{-2m-4}$, follows $z\in\Delta_{2m+4}$. The remaining case requires showing

$$\frac{\cos\alpha}{\cos(\alpha-\theta)} \leqq 1-\theta 2^{-2m-4}$$

for $(\alpha, \theta)\in T$. But the function

$$h(\alpha,\theta):=(1-\theta 2^{-2m-4})\cos(\alpha-\theta)-\cos\alpha$$

$$\times \text{has } h(\alpha, 0)=0,$$

$$h(\alpha,\alpha)\geqq 1-\alpha 2^{-2m-4}-\cos\alpha$$

$$=1-\frac{\alpha^3}{\pi^3}-\cos\alpha \geqq \frac{\alpha^2}{\pi}\left(1-\frac{\alpha}{\pi}\right)\geqq 0,$$

and $h_{\theta\theta}\leqq 0$ on T. Thus $h\geqq 0$ on T.

GEOMETRIC LEMMA 2. — *Let* $z=re^{i\theta}\in\Gamma_m$. There is an absolute constant C so that

$$\frac{|1-z|}{1-r}<C\,4^m, \tag{37}$$

and

$$\frac{|\theta|}{1-r}<C\,4^m. \tag{38}$$

There is an absolute constant $c>0$ so that if ρ is the radius of the largest circle centered at z and contained in Γ_{m+1}, then

$$\rho>c(1-r). \tag{39}$$

If, further, $z\notin\Gamma_{m-1}$, then

$$\frac{1-r}{1-r+|\theta|}<C\,2^{-m}. \tag{40}$$

Proof. — Let $\delta=1-r$, and without loss of generality assume $0\leqq\theta\leqq\pi$. Equation (40) can be rewritten $1/(1+(\theta/\delta))<C\,2^{-m}$ and so is equivalent to

$$\frac{\theta}{\delta}>C\,2^m. \tag{40'}$$

First let $z\in B_m=\{w \mid |w|<\cos\alpha\}$ where $\alpha=\pi/4\cdot 1/2^m$.
Then

$$\frac{|e^{i\theta}-z|}{1-r}<\frac{2}{1-r} \quad\text{and}\quad \frac{\theta}{1-r}\leqq\frac{\pi}{1-r},$$

but $1-r>1-\cos\alpha>C\alpha^2$ by (36) so (31) and (38) hold. For (39) note that ρ is greater than or equal to ρ^*, the radius of the largest disc about z contained in B_{m+1}. But

$$p^*=\cos\alpha_{m+1}-r \quad\text{and}\quad \frac{\cos\alpha_{m+1}-r}{1-r}$$

$$=\frac{\cos(\alpha/2)-1}{\delta}+1\searrow \quad\text{as } \delta\searrow 1-\cos\alpha \quad\text{or } r\nearrow\cos\alpha,$$

so to prove (39) for $z\in B_m$ it suffices to prove (39) for $z\in\partial B_m$.

If $z\in\Gamma_m\setminus B_m$, so that z is in the part of Γ_m near its apex 1, let $z^*=re^{i\theta^*}\in\partial\Gamma_m$ have $\theta^*>0$ and $|z|=|z^*|$. Then the left sides of (37) and (38) are both larger at z^* than at z. Apply the law of cosines to the triangle $(0, 1, z^*)$; noting that the side lengths are 1, $1-\delta$ and $|1-z^*|=:b$, and that $\cos(\pi/2-\alpha)=\sin\alpha$. We get $b^2-2(\sin\alpha)b+(2-\delta)\delta=0$, or since $b<\sin\alpha$ ($\overline{1P}$ has length $\sin\alpha$),

$$b=\sin\alpha-\sqrt{\sin^2\alpha-\delta(2-\delta)}=\frac{\delta(2-\delta)}{\sin\alpha+\sqrt{\sin^2\alpha-\delta(2-\delta)}}<\frac{(2-\delta)\delta}{\sin\alpha}<C\frac{\delta}{\alpha},$$

or

$$\frac{b}{\delta}<C.\frac{1}{\alpha}<C\,4^m,$$

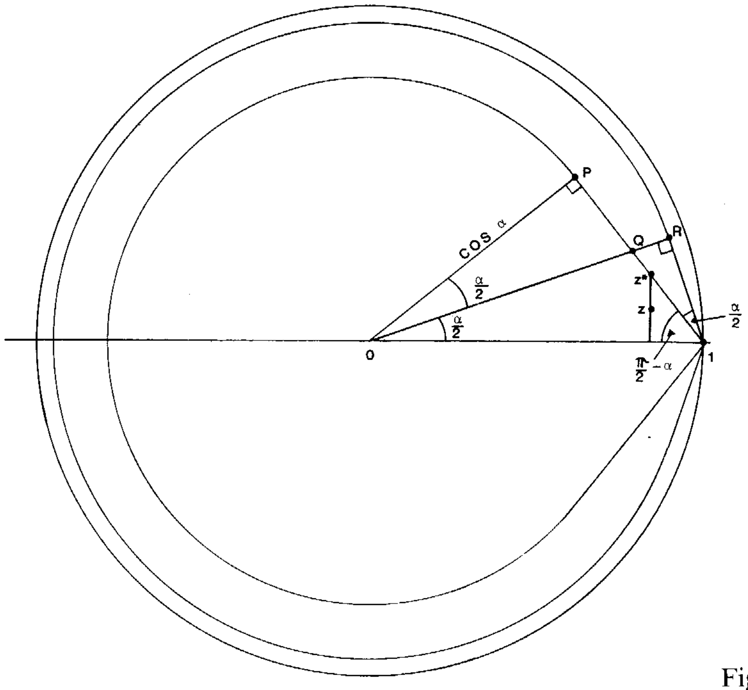


Fig. 2

which proves (37). For (38), apply the law of sines to the same triangle, getting

$$\frac{b}{\sin\theta^*}=\frac{r}{\sin((\pi/2)-\alpha)},$$

whence

$$\frac{\theta^*}{\delta}\leqq C\frac{r\sin\theta^*}{\delta}=C\frac{b\cos\alpha}{\delta}<C\frac{b}{\delta},$$

where $r\geqq\cos\alpha\geqq\sqrt{1/2}$ is used for the first inequality. Then (38) follows from (37).

As mentioned above, (39) for all $z\in B_m$ will follow from verifying (39) at $z=P$. (See *Fig.* 2.) If $z\in\overline{PQ}$, $\rho(z)=\cos\alpha/2-r$. We have

$$\nabla\left(\frac{\rho}{1-r}\right)=\nabla\left(\frac{\cos(\alpha/2)-r}{1-r}\right)=\frac{\partial}{\partial r}\left(\frac{\cos(\alpha/2)-r}{1-r}\right)e_r=-\left(\frac{1-\cos\alpha/2}{(1-r)^2}\right)e_r.$$

Since as z moves along PQ from P to Q r increases, the directional derivative of $\rho/(1-r)$ in the direction $\overrightarrow{PQ}$ is negative. Thus $(\rho/(1-r))(P)>(\rho/(1-r))(Q)$. Further if $z\in\Gamma_m\setminus B_m$, then the fraction $(\rho/(1-r))(z^*)$ has a smaller numerator than, but the same denominator as, $(\rho/(1-r))(z)$. In conclusion, (39) will follow from $(\rho/(1-r))(z)>c$ for all $z\in\overline{Q\,1}$.

Let now $z\in\overline{Q\,1}$. Let S be the closest point of $\overline{R\,1}$ to z so that $\overline{zS}\perp\overline{S\,1}$ and $\rho(z)=|\overline{zS}|$. Looking at the triangle $(z, S, 1)$ shows that

$$\rho(z)=|\overline{z\,1}|\sin\frac{\alpha}{2}=b\sin\frac{\alpha}{2}$$

$$=\frac{\delta(2-\delta)\sin\alpha/2}{\sin\alpha+\sqrt{\sin^2\alpha-\delta(1-\delta)}}>\frac{\delta\cdot 1\cdot\sin\alpha/2}{2\sin\alpha}=\frac{\delta}{4\cos\alpha/2}\geqq\frac{\delta}{4}.$$

This proves (39).

For (40′) we will show $\theta/\delta>c\,2^{m+1}$ for all $z\in\Gamma_{m+1}/\Gamma_m$. (The index shift allows us to use figure 2.) If $\alpha/2\leqq\theta\leqq\pi$, $r\geqq\cos\alpha$ and

$$\frac{\theta}{\delta}\geqq\frac{(\alpha/2)}{1-\cos\alpha}>C\,2^{m+1}$$

by (36). If $0<\theta<\alpha/2$, z is inside triangle $(1, Q, R)$. So if $z^t:=\overline{Q\,1}\cap\overline{OZ}$, then $\theta(z^t)=\theta(z)$ while $\delta(z^t)>\delta(z)$. Hence we need only deal with the case of $z\in\overline{1\,Q}$. As in the proof of (38) above, for such a z we have

$$\frac{\theta}{\delta}\geqq C\frac{r\sin\theta}{\delta}=C\frac{b\cos\alpha}{\delta}$$

$$=C\frac{\cos\alpha}{\delta}\cdot\frac{\delta(2-\delta)}{\sin\alpha+\sqrt{\sin^2\alpha-\delta(2-\delta)}}$$

$$>C\cos\alpha\frac{1}{2\sin\alpha}>C\,2^{m+1}.$$

This proves (40′), hence (40), and hence geometric lemma 2.

Remark 1. – In the proof of this lemma, we tried to avoid "geometric intuition." Only strict dominations and equalities were used, except for the intuitive estimates

$$\sin x < C x \qquad \text{and} \qquad \frac{1}{\sin x} < C \frac{1}{x},$$

which can be made precise by using elementary calculus to show the functions $x - \sin x$ and $\sin x - (2/\pi) x$ nonnegative on $[0, \pi/2]$.

Remark 2. – The constant 4^m in (37) and (38) is sharp at certain points far away from the apex of Γ_m. Near $z = 1$, the above proof shows that 2^m will do.

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