# Review of Steven Krantz's *A Panorama of Harmonic Analysis*

by J. Marshall Ash
Department of Mathematics, DePaul University, Chicago, IL 60614

March 17, 2000

I had some trouble with the topology section of the University of Chicago's master of science exam in May 1963. But the analysis section went very well. No small part of the credit for this was due to Antoni Zygmund who had taught me real analysis I and II and to Alberto Calderón who had taught me complex analysis I and II. In very short order, I abandoned my plan of specializing in point set topology and picked Fourier series as the main topic for my next hurdle, the two topic exam. Over the next year, Bill Connant, Larry Dornoff, and I read Zygmund's *Trigonometric Series* in preparation for the exam. After the exam, Professor Zygmund accepted me as his student and my career as a harmonic analyst began.

Looking through Krantz's *A Panorama of Harmonic Analysis* feels like watching a home video production entitled "The Zygmund School of Analysis: 1965-1999." Throughout this period, I was lucky enough to be near the University of Chicago, where the Monday 3:45 PM Calderón-Zygmund seminar has featured, among many other things, just about every development in harmonic analysis mentioned in Krantz's book. (From 1967 to 1969 I was away from Chicago, but during that time Eli Stein's seminar at Princeton University served a similar function for me.) These were exciting times in harmonic analysis and my connection with the University of Chicago's seminar placed me near the center of the action.

I have always been attracted by questions that have crisp, easily grasped statements. A good example of such was Lusin's conjecture that there could exist a real-valued square integrable function defined on the interval $\mathbb{T} = [0, 2\pi)$ whose Fourier series diverged at each point of a set of positive measure. Already in 1927 Kolmogorov had given an example where the function was integrable, but not square integrable and the Fourier series diverged at every point. Since giving an example seemed like it couldn't be very hard, I proposed to Zygmund that I take the Lusin conjecture for my thesis problem. He immediately discouraged this idea, explaining that this problem might prove to be rather difficult. Zygmund had realized that the square integrable case was much deeper than the integrable case, even though his almost infallible intuition this time predicted the existence of an example. In the early fall of 1964 my fellow graduate student

Lance Small came back from a summer visit to Berkeley carrying the news that Lennart Carleson had just proved that the Fourier series of a square integrable function actually converges almost everywhere. He was closely questioned by Zygmund and Calderón who thought that he could not have gotten the story straight. But he had and Carleson had. In other words, Carleson had resolved the Lusin conjecture and Small had made a faithful report of what Carleson had done. Several years later when I spent two months working through Richard Hunt's careful exposition of his extension of Carleson's theorem, it became clear to me that although Zygmund's guess about the outcome of the conjecture had been wrong, his assessment that an extremely high level of mathematics would be required to decide the issue was quite accurate.

After finishing my degree in 1966, I was a Ritt instructor at Columbia. I was at a loss for how to begin my career as a research mathematician. My inertia was assisted by the cultural cornucopia that New York City provided, and also by the Columbia campus protest movement featuring Mark Rudd, SDS, and the occupation of the math building which contained my office. I wrote to Zygmund. He suggested that I get into partial differential equations. This certainly proved prophetic. The great bulk of harmonic analysis being done now seems to be in connection with work in partial differential equations. One of the ways to see this is to note that the preponderance of talks being given nowadays at the Calderón-Zygmund seminar fits this profile. Nevertheless, most of my own interest never did move in that direction.

One thing I did to stay mathematically alive was to attend Stein's seminar at Princeton. One of the talks I heard there was given by Stein's extremely young Ph.D. student Charles Fefferman. At the time, I did not have a sufficient overview of the fundamentals of harmonic analysis to appreciate the depth and beauty of his mathematics, but fortunately I have heard him lecture many times since. It is also fortunate that his expository skills have improved from very good to extraordinary. For example, I consider it a high compliment when I say that Krantz's book does justice to the lectures I later heard in Chicago wherein Fefferman explained the proof of his theorem that the characteristic function of the unit ball is not a multiplier on $L^p\left(\mathbb{R}^2\right)$ when $p \neq 2$.

After three years at Columbia, I moved to DePaul and back to the Calderón-Zygmund seminar. When I first arrived in Chicago, the harmonic analysts there were reading Igari's book on multiple Fourier series.[I] Grant Welland and I immediately began doing research in this direction, and the main thrust of my mathematical career has been in this direction ever since. For this reason I have an especially strong interest in chapter 3 of *A Panorama of Harmonic Analysis* which is entitled *Multiple Fourier Series.*

Extending the work of Carleson, Richard Hunt proved that the Fourier series of an $L^p(\mathbb{T})$ function converges almost everywhere, provided that $p > 1$. What happens in dimension two? Krantz points out that if "converges" means that the partial sums include terms of the series with indices lying in the dilates of a fixed polygon, the analogue of Hunt's Theorem is true, whereas if "converges" means that the partial sums are taken to include the terms with indices lying in rectangles of variable eccentricity, then there is a counterexample, due to

Charles Fefferman. (Larry Gluck and I later added a small "bell and whistle" to that example.) But the most important question of what happens when "converges" means that the partial sums include the terms with indices lying in the dilates of an origin-centered disk remains unsolved. Fefferman's Theorem that the unit ball is not a multiplier guarantees that it is not enough for $p$ to be greater than 1, but gives no insight as to what happens when $p = 2$. This leaves open the question of whether the Fourier series of an $L^2\left(\mathbb{T}^2\right)$ function has circularly convergent partial sums almost everywhere. To my way of thinking, this question is the Mount Everest of multiple Fourier series.

An interesting question not dealt with in chapter 3 is the question of uniqueness. Is the trigonometric series with every coefficient equal to zero the only one that converges at every point to 0? I have spent much of my life working on this question and have been pleased to see an almost complete set of answers discovered.[AW] The only thing I want to say here is that uniqueness has been shown to hold in many cases, but here the situation is opposite to that for convergence of Fourier series mentioned above. We *do* know that uniqueness holds for circularly convergent double trigonometric series, but we *don't* know if it holds for square convergent double trigonometric series.

Speaking of chapter 3, one thing I would like to clarify is the definition of restricted rectangular convergence. I tried to explain this very subtle definition in my 1971 paper with Welland and I will take another try at it here. Fix a large number $E \gg 1$. Let $\{a_{mn}\}_{m=1,2,\ldots;n=1,2,..}$ be a doubly indexed series of complex numbers and denote their rectangular partial sums by $S_{MN} = \sum_{m=1}^{N}\sum_{n=1}^{N} a_{mn}$. Then say that $S = \sum\sum a_{mm}$ is $E$-restrictedly rectangularly convergent to the complex number $s\left(E\right)$ if

$$\lim_{\substack{M,N\to\infty \\ \frac{1}{E} < \frac{M}{N} < E}} S_{MN} = s\left(E\right).$$

Finally say that $S$ is restrictedly rectangularly convergent if there is a single complex number $s$ such that for every $E$, no matter how large, $s\left(E\right)$ exists and is equal to $s$. An example may help to clarify this. For $n = 2, 3, ...$, let $a_{n^2,1} = n, a_{n^2,n} = -n$, and let $a_{mn} = 0$ otherwise. Notice that $S_{MN} \neq 0$ only if there is an $n > N$ such that $n^2 \leq M$ so that $a_{n^2,1}$ is included in the partial sum, while $a_{n^2,n}$ is not. But then $N^2 < n^2 \leq M$ so that $M/N > N$. Thus if any eccentricity $E$ is given, as soon as $N$ exceeds $E$, the condition $M/N < E$ becomes incompatible with $S_{MN} \neq 0$. In other words, $s\left(E\right)$ is 0 for every $E$ so that this series is restrictedly rectangularly convergent to 0. And this happens despite the fact that $S_{N^2,N-1} = N$, so that $\lim_{\min\{M,N\}\to\infty} S_{MN}$ does not exist, which is to say that $S$ is not unrestrictedly rectangularly convergent.

Krantz has made wonderful selection choices for all of his chapters. In particular, I think that ending with a chapter on wavelets represents a very good estimate of which way a good part of the winds of harmonic analysis are blowing right now as well as a shrewd guess as to which way they will blow in the near future. A botanist recently asked me for some help in finding a good mathematical representation for ferns that she has been studying. Although my work is usually not very applied, I have looked into this a little bit and it seems likely that wavelets may prove to be the way to go here.

*A Panorama of Harmonic Analysis* is Carus Mathematical Monograph number 27. The Publisher, the Mathematical Association of America say that books in the series "...are intended for the wide circle of thoughtful people familiar with basic graduate or advanced undergraduate mathematics...who wish to extend their knowledge without prolonged and critical study of the mathematical journals and treatises." Krantz has done an admirable job of carrying out the publisher's intentions. The right way to read this book is quickly, without too much fussing over the details. While other books, such as those by Zygmund[Z] and Stein and Weiss[SW], are probably better for a graduate student who will need to achieve technical competence in the area, I can't think of a more efficient way to obtain such a well-balanced overview of the entire subject.

## References

[AW] J. M. Ash and G. Wang, *A survey of uniqueness questions in multiple trigonometric series,* A Conference in Harmonic Analysis and Nonlinear Differential Equations in Honor of Victor L. Shapiro, Contemporary Math., **208**(1997), 35–71.

[I] S. Igari, *Lectures on Fourier Series in Several Variables,* University of Wisconsin, Madison, 1968.

[SW] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press, Princeton, 1971.

[Z] A. Zygmund, *Trigonometric Series,* 2nd rev. ed., Cambridge Univ. Press, New York, 1959.