MULTIPLE TRIGONOMETRIC SERIES

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1. INTRODUCTION

The purpose of this article is to give the reader a feeling for the obstacles that arise when one tries to extend the far-reaching and well-developed theory of one dimensional trigonometric series to higher dimensions. I readily confess to slanting this talk towards areas I have worked in and make no claim of comprehensiveness. Furthermore, I will resist the temptation of stating a best known or "best possible" result whenever a less good result is easier to understand but still captures the spirit of the situation. (For example, the hypothesis of $f \in L^p$ might be used when a weaker hypothesis such as $f \in L(\log^+ L)^2$ would be sufficient.) In such cases, I will try to give enough references so that the interested reader can trace down the stronger version of the result.

I will mainly explicate the theory of double Fourier series, whose generality and difficulty is intermediate between the *terra firma* of one dimension and the *terra incognita* of three dimensions. At the present time, the passage from two to three dimensions seems far more substantial and non-trivial than that from one to two or than that from three to more.

Let $f(x, y)$ be a measurable complex valued function defined on the unit square $T^2 = [-\frac{1}{2}, \frac{1}{2}) \times [-\frac{1}{2}, \frac{1}{2})$.

If the Lebesgue integral $\iint_{T^2} |f(x, y)|^p \, dx \, dy$ (where $p \geq 1$) is finite, we say $f \in L^p$. In this case, f has a Fourier series given at each (x, y) of T^2 by

$$S[f](x, y) = \sum_{m, n} \hat{f}_{mn} e^{2\pi i(mx + ny)}, \tag{1.1}$$

where the Fourier coefficients $\hat{f}_{mn}$ are given by

$$\hat{f}_{mn} = \iint_{T^2} f(s, t) e^{-2\pi i(ms + nt)} \, ds \, dt, \tag{1.2}$$

and the summation is indexed by the m-n lattice plane—the set of all ordered pairs of integers. This Fourier series is a very special case of a double trigonometric series where the $\hat{f}_{mn}$ are replaced by arbitrarily chosen complex constants c_{mn}. As in the one dimensional case, the set of all double Fourier series is only a very, very, tiny subset of the set of all double trigonometric series.

The lattice plane has no natural ordering and many important trigonometric series are not absolutely convergent, so the first order of business is to determine an ordering.

We will take the terms corresponding to the four points (m, n), $(m, -n)$, $(-m, n)$, and $(-m, -n)$ together (just as one takes $\hat{f}_n e^{2\pi inx} + \hat{f}_{-n} e^{-2\pi inx}$ as a term in the one dimensional theory). Even this is less natural than in the one dimensional case as our later discussion of the failure of Plessner's theorem will indicate.

This grouping of the four terms reduces the problem of ordering the lattice plane to that of ordering the lattice quadrant. How shall we do this? The answer is far from clear. In fact, there very probably is no one answer.

To get a feel for the situation we will consider some numerical series that ought not to converge, but which do with respect to some of the usual orderings.

Let $s_{m,n} = \sum_{(i,j) \text{ southwest of } (m,n)} a_{ij}$ be the rectangular partial sums.

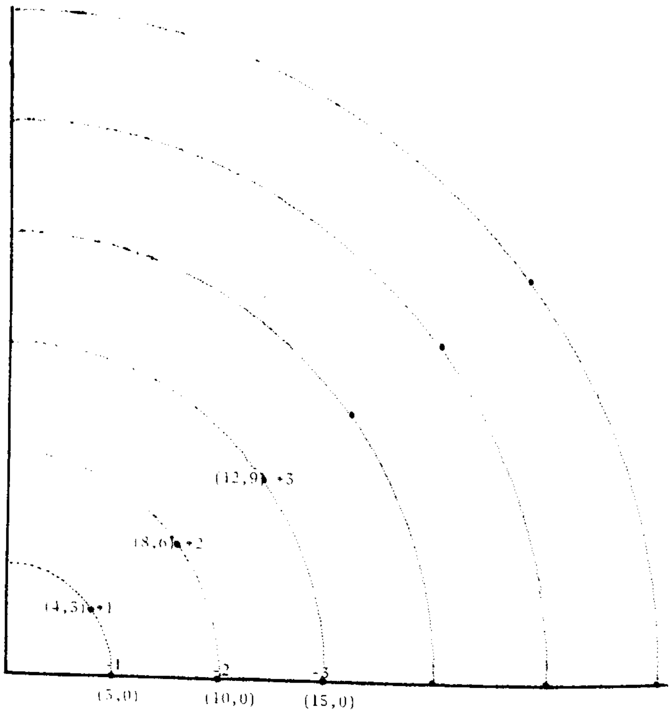

FIG. 1

For example,

$$s_{3,1} = a_{0,1} + a_{1,1} + a_{2,1} + a_{3,1} + a_{0,0} + a_{1,0} + a_{2,0} + a_{3,0}.$$

Example 1. Let $a_{5k,0} = -k$, $a_{4k,3k} = k$, $a_{mn} = 0$ otherwise (Figure 1).

A numerical double series is *square convergent* if the limit of s_{nn} exists as n tends to infinity. Here $s_{5k-1,5k-1} = k$ so $\lim_{n\to\infty} s_{nn}$ doesn't exist. This series is not square convergent. Let $s_r = \sum_{\sqrt{m^2+n^2}<r} a_{mn}$. Then a numerical series is *circularly convergent* if the limit of s_r exists as r tends to infinity. However, in example 1, for all r, $s_r = 0$, so $\lim_{r\to\infty} s_r = 0$, so the series is circularly convergent.

Example 2. Let $a_{kk} = -k$, $a_{k0} = k$, $a_{mn} = 0$ otherwise.

Here $s_{nn} = 0$ for all n so the series is square convergent. (Draw a picture.)

A third method of convergence is restricted rectangular convergence. Here we are interested in the "limit" of the not too eccentric rectangular partial sums—rectangles that are "sort of" close to squares. Since the set of all such rectangles is not linearly ordered the definition requires some fancy footwork. We say that the series *converges restrictedly rectangularly* to s if for every fixed $E > 1$, $\sup_{(m,n)\in W_E(N)} |s_{mn} - s| \to 0$ as $N \to \infty$, where $W_E(N)$ $(= \{(m,n) : m \geqslant N, n \geqslant N, Em \geqslant n \geqslant E^{-1}m\})$ are a sequence of wedges cut out of the lattice quadrant shown in Figure 2.

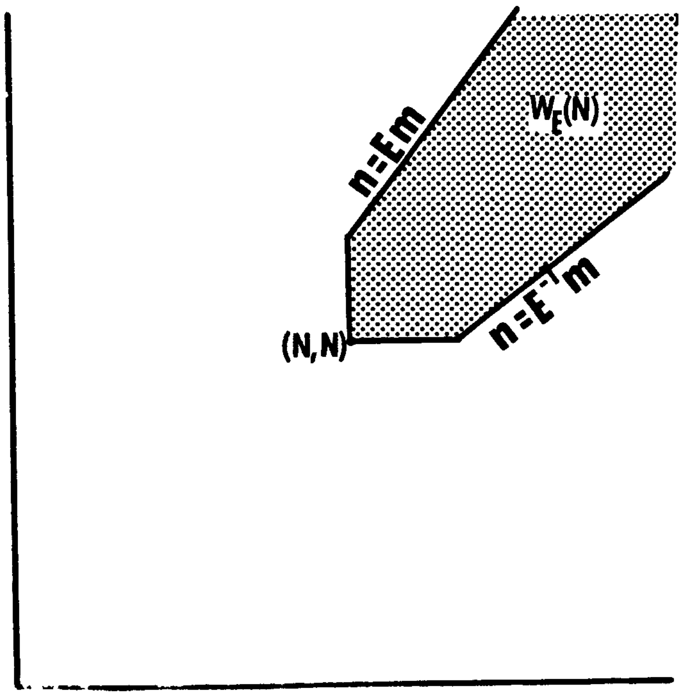

FIG. 2

The intuitive idea behind this picture is that if E is not too much bigger than 1, then a rectangle with southwest corner at (0, 0) and northeast corner in $W_E(N)$ is pretty nearly a square. Furthermore, the point of the N is that as N becomes very large you are looking at a "late" partial sum.

It is equivalent to demand that $s_{m_i, n_i} \to s$ for every sequence (m_i, n_i) tending to (∞, ∞) in such a way that the two ratios m_i/n_i and n_i/m_i remain bounded.

Observe that example 2 is not restrictedly rectangularly convergent since $s_{k, k-1} = k$.

The last method of convergence I will discuss is that of *unrestricted rectangular convergence*. We say $s_{mn} \to s$ unrestrictedly rectangularly if $\lim_{\min(m, n) \to \infty} s_{mn} = s$, i.e., if $\sup_{m, n \geqslant N} |s_{mn} - s| \to 0$ as N increases.

Example 3. Let $a_{k^2, 0} = k$, $a_{k^2, k} = -k$, $a_{mn} = 0$ otherwise.

This example converges (to 0) restrictedly rectangularly, but fails to converge unrestrictedly rectangularly since $s_{k^2, k-1} = k$. (Draw a picture.)

Example 4. Let $a_{k, 0} = k$, $a_{k, 1} = -k$, $a_{mn} = 0$ otherwise.

This is a nasty series—it causes a lot of trouble. Here the rectangular sums, as soon as they are at least $n \times 2$, are zero. So the series Σa_{mn} is unrestrictedly rectangularly convergent to zero. Nevertheless, it has some pretty horrible partial sums. Just think about any partial sum involving only the bottom line $s_{n0} = \Sigma_{i=0}^{n} a_{i0} = n(n+1)/2$. So this series has terribly bad partial sums; it has big terms, and yet it is unrestrictedly rectangularly convergent. (Incidentally, it is, of course, circularly divergent.)

The relations between modes of convergence for two dimensional series can be visualized by Figure 3.

In Figure 3, $A \to B$ means that convergence of a numerical series with respect to method A forces convergence of that series with respect to method B; while $C \nrightarrow D$ means there is a series converging with respect to method C and diverging with respect to method D. These introductory remarks already show a separation between the one and two dimensional situations—things will usually remain worse for the two dimensional case than for the one dimensional case when we throw some exponentials in and begin our study of trigonometric series.

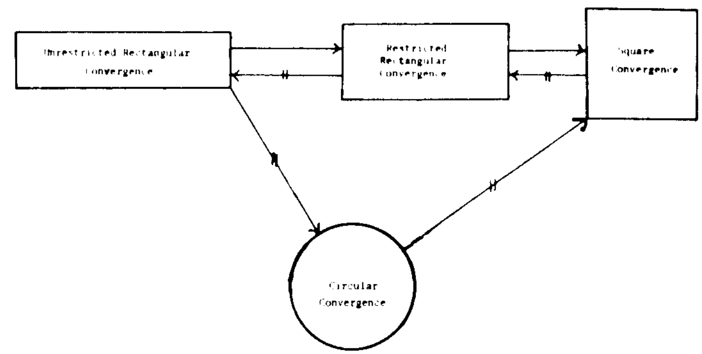

FIG. 3

The main body of my discussion will be broken into five areas. They will be convergence and divergence of Fourier series, the effect of convergence on the size of coefficients, localization, the relative behaviors of a trigonometric series and its conjugates, and the uniqueness of the representation by trigonometric series. In each area, I will very quickly run over the main points of the one dimensional theory, and then tell you how far the results can be extended into higher dimensions.

2. CONVERGENCE AND DIVERGENCE OF TRIGONOMETRIC SERIES

Here the one dimensional theory is pretty clear. The positive result is the

CARLESON-HUNT THEOREM: *If $f \in L^p(T)$, $p > 1$, then $S_n[f](x)$ converges at almost every point x.* (2.1)

Carleson did the $p = 2$ case in 1965 and Hunt extended to all $p > 1$ in 1967 [4], [12]. I want to mention two negative results. The first is the very well-known counterexample of A. N.

Kolmogorov:

There is a function k in $L^1(T)$ *whose Fourier series diverges everywhere.* (2.2)

The other negative result, which is due to J. Marcinkiewicz, is possibly not quite as familiar. It is

There is a function $m \in L^1(T)$ *which is finitely oscillating and divergent almost everywhere* [18]. (2.3)

Finitely oscillating means that if we fix a point x and look at the partial sums $s_n = S_n[f](x)$, they wiggle (i.e., $\lim\sup s_n - \lim\inf s_n > 0$) but they do not go off to infinity (i.e., there is a finite number $M(x)$ such that $\sup |s_n| < M$).

Kolmogorov in 1923 produced a weaker counterexample of an L^1 function with series divergent only almost everywhere [15]. (This means that there was an exceptional set of zero Lebesgue measure on which convergence might occur.) Three years later, he was able to construct the perfectly divergent function of (2.2) above [16]. We emphasize this point because the "almost" of (2.3) cannot be dropped. For suppose there were an $L^1(T)$ function $m_1(x)$ finitely oscillating at each x. Then the union of the closed sets $E_N = \{x \in T : \sup_n |S_n[m_1](x)| < N\}$ as N ranges over the positive integers would be all of T, so that by the Baire category theorem some E_{N_0} would contain an open interval (a, b) in which all the $S_n[m_1](x)$ and so also $m_1(x)$ would be bounded by N_0. (Recall m_1 is the $(C, 1)$ limit of its partial sums almost everywhere.) Hence replacing m_1 by 0 outside (a, b), we would obtain a new function m_2 bounded on T and thus by (2.1) convergent almost everywhere on T. But $m_1 \equiv m_2$ on (a, b) so that by localization (see section 4 below) m_2 must diverge almost everywhere on (a, b)—a contradiction.

In two dimensions, the answer to the question of convergence varies with the method of summation. To start with let's look at restricted rectangular convergence. There is a tremendously good function f with an everywhere divergent Fourier series. How good

is f? This good:

(i) f is continuous,
(ii) f has a Fourier series of power series type, (2.4)
(iii) f has everywhere uniformly bounded rectangular partial sums.

Condition (i) is much stronger than $f \in L^p$, so we have a marked contrast to (2.1). By condition (ii) we mean that in f's Fourier expansion (see 1.1) the coefficients $\hat{f}_{mn}$ are zero if (m, n) is not in the first quadrant. The one dimensional analogue of this—$\hat{f}_m = 0$ if $m < 0$—often improves things greatly; it doesn't seem to help much here. The impossibility of dropping the "almost" from the conclusion in Marcinkiewicz's example (2.3) shows that condition (iii) is in very dramatic contrast to the one dimensional case. If, however, you prefer your examples more divergent, it is easy to change f so that (iii) is replaced by "f has $\lim\sup_{n\to\infty} |S_n[f](x)| = \infty$ everywhere". This example is essentially due to Fefferman in 1970 [10] with a few of the frills added in [1].

You can get positive results if you shift gears by changing the method of convergence. Recall from our discussion of numerical series that it would seem easier for a series to be square convergent; and, sure enough, it is a lot easier. Here is a theorem which indicates this:

If $f \in L^p$, $p > 1$, *then* $S_n[f] \to f$ *almost everywhere.* (2.5)

This was proved around 1970 simultaneously by Fefferman (in the United States), Sjolin (in Sweden), and Tevzadze (in the USSR) [9], [22], [25]. The method of proof is quite interesting. We would like to do induction starting from Carleson's theorem (2.1) but for technical reasons this doesn't work. So we introduce a "butterfly". If

$$f \sim \sum \sum \hat{f}_{mn} e^{2\pi i(mx+ny)},$$

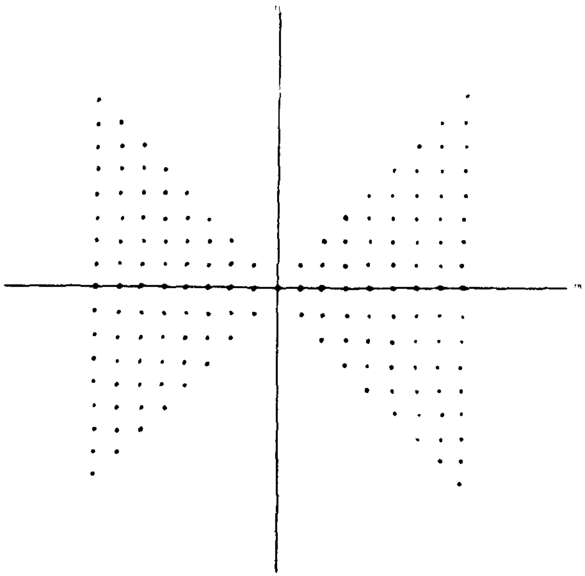

FIG. 4

decompose the sum into

$$\sum_{m=-\infty}^{\infty} \sum_{n=-|m|}^{|m|} + \sum_{m=-\infty}^{\infty} \sum_{|n|>|m|} . \tag{2.6}$$

To both terms correspond functions which are also in L^p (M. Riesz's theorem) and for both these functions the lattice plane regions being summed over are now butterflies (e.g., the first summand's index set looks like Figure 4), and the technical obstacle to the induction vanishes, as is easily seen.

I will mention in passing that for double Walsh-Fourier series it is not clear if both of the sums in (2.6) represent L^p functions if $p \neq 2$, so the analogue of theorem (2.5) for double Walsh series follows easily only for $p = 2$. If $p < 2$, the question remains open.

For circular convergence there are some negative results but there is still an open question. Here if $f \in L^p$ with $p < 2$, circular divergence may occur. This follows from Fefferman's counterexample for the multiplier problem [7], [8]. If $p = 2$, the question remains open. I think this is a nice question to work on.

There are various ways to clean up the counterexample (2.4). One thing we can do is to put more stringent restrictions on the function f. For example, we might demand that $f \in L^2(T^2)$ and that one of its partial derivatives exist and also belong to $L^2(T^2)$ [22]. A second approach is to work with summability instead of convergence. For example, for f in L^p the $(C, 1, 0)$ means of the Fourier series—

$$\sigma_{mn} = \frac{1}{m+1}(S_{0n} + S_{1n} + \cdots + S_{mn})$$

do converge to f unrestrictedly rectangularly almost everywhere [3]. (We were able to get away with averaging only in the m direction because of Carleson's theorem.)

3. CONVERGENCE AND GROWTH OF COEFFICIENTS

The point here will be to see how bad coefficients can be for a convergent trigonometric series.

The one dimensional situation is very nice. We assume that $T = \sum a_n e^{2\pi inx}$ converges on the set E, where $|E|$—the Lebesgue measure of E—is greater than 0. By definition this means that

$$a_n e^{2\pi inx} + a_{-n} e^{-2\pi inx} \to 0, \qquad x \in E, |E| > 0. \tag{3.1}$$

The Cantor-Lebesgue theorem postulates (3.1) and concludes that

$$a_n \to 0 \quad \text{as} \quad |n| \to \infty.$$

So convergence on a set of positive measure forces the coefficients to go to 0. Again, the answer as to whether this works in higher dimensions depends on the method of convergence. We will assume throughout this section that a double trigonometric series $T = \sum a_{mn} e^{2\pi i(mx+ny)}$ is convergent (under the mode of convergence being discussed) at each $x \in E$, where $|E| > 0$.

For circular convergence the conclusion is very nice. Not only do the coefficients go to zero, but, in fact, the sum of the squares of the coefficients lying on a circle

$$\sum_{\{(m,n):\, m^2+n^2=r^2\}} |a_{mn}|^2$$

tends to 0 as the radius r tends to ∞. This was first proved when E has full measure by Roger Cooke in 1971 and the hypothesis of $|E| = 1$ was weakened to $|E| > 0$ by Antoni Zygmund in 1972 [6], [28]. The partial sums are no problem because we are dealing with a one-parameter family. For one parameter methods (e.g., circular —which is indexed by r—and square—which is indexed by (n, n)) convergence implies that the partial sums tend to a limit by definition.

Let us now pass to the unrestricted rectangular case. Here convergence doesn't quite force the coefficients to go to 0. It does force

$$a_{mn} \to 0 \text{ in the northeast.}$$

We can prove this as fast as we can explain what is meant. Fix $(x, y) \in E$ and take a rectangular partial sum $S_{m,n} = S_{m,n}(x, y)$ which is close to the limit (this will be true if m and n are both big); add $S_{m-1,n-1}$; then subtract $S_{m-1,n}$ and $S_{m,n-1}$. All 4 of these partial sums are near the limit, so since 2 are taken positive and 2 are taken negative the resultant expression is small. But a moment's thought will show that this expression has all terms 0 except for the (m, n) one:

$$a_{m,n}e^{2\pi i(mx+ny)} + a_{m,-n}e^{2\pi i(mx-ny)} + a_{-m,n}e^{2\pi i(-mx+ny)} + a_{-m,-n}e^{-2\pi i(mx+ny)}. \tag{3.2}$$

This argument, which is nothing more than the two dimensional version of $a_n = \sum_{\nu=1}^{n} a_\nu - \sum_{\nu=1}^{n-1} a_\nu$, is what we call the Mondrian proof. (See [3, p. 411] for a picture and further details.) Finally, there is an easy inductive extension of the Cantor-Lebesgue theorem that deduces from (3.2) that the coefficients themselves must be small. But notice that the reasoning that made (3.2) small required *both* m and n to be large—that is (m, n) to be in the northeast.

Example 4 above might seem to contradict this proof. What's going on, of course, is that the terms are not in the northeast; rather they are all due east—that is, down on the m axis. It looks as though you have no control at all over what happens in the east. However, it turns out that using a very clever lemma of Paul J. Cohen, you can prove that the coefficients are all bounded regardless of where they are:

$$|a_{mn}| \leq M \text{ for some } M \text{ and all } (m, n),$$

[3, p. 410]. If we keep thinking about numerical example 4 we can see that this is a little bit surprising. Near the end of section 1, we said that double trigonometric series were usually just as bad as double numerical series. Here is one case where they are better. We can use the positive measure of the set E to disallow possibilities like those of example 4.

Square convergence is awful. The series can square converge on a set of positive measure, even on a set of full measure and still have incredibly big coefficients. Here is the example:

$$T = \sum_{n=1}^{\infty} n^{10^6}(\sin^2 \pi x)^n e^{2\pi i n y}. \tag{3.3}$$

This looks like a single series—we can see an x and a y in it but there is only one summation sign. To see what's going on, I'll change the sines into exponentials via Euler's formula:

$$T = \sum \frac{n^{10^6}}{(2i)^{2n}} \left(e^{\pi i x} - e^{-\pi i x}\right)^{2n} e^{2\pi i n y}.$$

If we apply the binomial theorem, we find

$$\left(e^{\pi i x} - e^{-\pi i x}\right)^{2n} = e^{2\pi i n x} + \cdots + (-1)^n \binom{2n}{n} + \cdots + e^{-2\pi i n x} \tag{3.4}$$

so that the n-th term of the summation actually includes terms associated to the lattice points on the "up" butterfly wing at height n:

$$(-n, n), (-n+1, n), \ldots, (0, n), \ldots, (n, n).$$

Thus the partial sums of the single series (3.3) are exactly the square partial sums of T thought of as a double trigonometric series; so if we want to know whether T is square convergent we simply have to ask whether the single series (3.3) is convergent. Well you can tell we're up to tricks with the n^{10^6} term. On the other hand the size of the n-th term is being driven down much faster, in fact geometrically by the $(\sin^2 \pi x)^n$ term. We're in trouble only if $\sin^2 \pi x = 1$ and this occurs only on the extreme edge of T^2, that is, only on the line $x = -\frac{1}{2}$. So T is convergent on $(-\frac{1}{2}, \frac{1}{2}) \times [-\frac{1}{2}, \frac{1}{2})$, which is certainly almost everywhere. Now how about the coefficients? This sounds like a horribly messy job, but it's not. In fact, we'll take only one term from the expansion (3.4)—the middle term $(-1)^n\binom{2n}{n}$—which corresponds to the coefficient $a_{0,n}$. To avoid worrying about i's and minus signs we'll look only at absolute values. We have

$$|a_{n,0}| = \frac{n^{10^6}}{2^{2n}}\binom{2n}{n}.$$

From Stirling's formula it easily follows that

$$\frac{1}{2^{2n}}\binom{2n}{n} \cong \frac{1}{\sqrt{\pi n}}$$

so that

$$|a_{n,0}| \cong \frac{1}{\sqrt{\pi}} n^{999,999.5},$$

which is quite unbounded; coefficients of square convergent series can be pretty bad. (See [3, p. 408] for details.)

Recall the diagram at the end of the introduction which summarized the incompatibility of the four convergence methods for numerical series. Examples such as 3.3 allow us to construct a similar diagram in which, for example, "$A \nrightarrow B$" means "There is a trigonometric series T converging at each point of some set E of positive measure by method A, but not converging at any point of E by method B". All the arrows (and non-arrows) are the same except that it *may* be that the circular convergence of T on a set forces the square (or perhaps even the unrestricted rectangular) convergence of T at almost every point of that set. This is an open question [3, p. 420].

It is interesting to compare the results of sections 2 and 3 for the various methods of summation. In section 2 we wanted a function's goodness to force its Fourier series convergence. Since it is very easy for series to square converge, in section 2 one gets the best theorem for square convergence—in fact one gets essentially no theorem at all for the other rectangular methods. Conversely in section 3 the assumption is convergence and the hoped for conclusion is good behavior of the series coefficients. Here since unrestricted rectangular convergence is the most difficult of the rectangular methods, it provides the strongest hypothesis and hence the best theorems, while square convergence yields the poorest conclusions. In short, which method of summation is best depends on what you're trying to do.

4. LOCALIZATION

We again start with the one dimensional case, letting our function f belong to $L^1(T^1)$. If $f = 0$ in a neighborhood of the point x; then, regardless of how bad f is anywhere else, its Fourier series converges to 0 at x. This phenomenon is called localization. In other words, the behavior of the partial sums of the Fourier series only depends on how the function looks right near the point.

Now for 2 variables and rectangular methods, localization fails. In fact, there is an f with a differential at each point (so that in particular at each (x, y) $\partial f/\partial x$ and $\partial f/\partial y$ exist), a point (x_0, y_0), and neighborhood N of (x_0, y_0) on which $f \equiv 0$, for which even the square partial sums (and hence *a fortiori* the other two types) get out of hand:

$$\sup_n |S_{n,n}[f](x_0, y_0)| = \infty.$$

For circular convergence localization fails again. I'm not sure if you can do it with a differentiable function (I wouldn't be surprised if you could), but I know it fails with a continuous function [13], [14].

So localization is a complete failure—the exact analogue of the one dimensional result is simply false.

Well, as usual there are a lot of ways around the problem—at least three. One is to demand that f be very, very smooth. For example, we can demand that the two partial derivatives be themselves L^1 functions—i.e., that f belong to the so-called Sobolov space $W_{1,1}$. There is a 1972 work of Goffman and Liu [11] and another paper of Liu (also in 1972 [17]) on this.

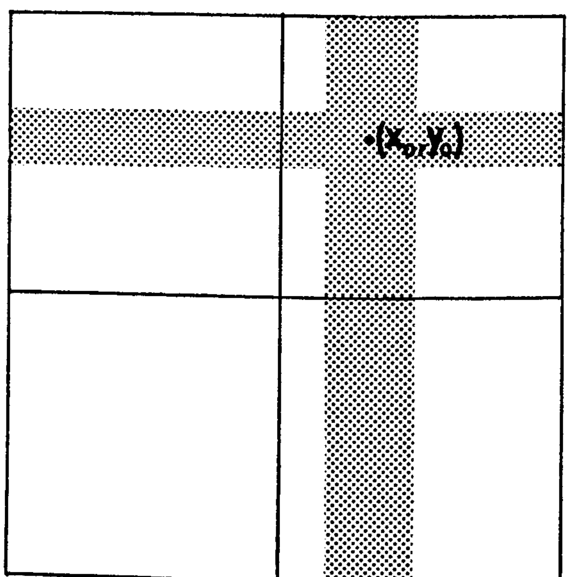

Fig. 5

A second way out is to make f more zero. If it's not 0 enough already, make it more 0—make it 0 on an entire cross-neighborhood of (x_0, y_0). We won't define a cross-neighborhood but it looks like Figure 5. The cross may be very thin but it has to go all the way to the boundary of T^2 in all 4 directions. If $f = 0$ on such a neighborhood, then you will get localization for all the rectangular methods.

A third way is to make f a little bit good—say continuous—and to replace convergence by summability. For example, the $(C, 1, 1)$ partial sums

$$\frac{1}{m+1}\,\frac{1}{n+1}\sum_{i=0}^{m}\sum_{j=0}^{n} S_{ij}[f](x_0, y_0)$$

will converge to 0 when f is 0 on an (ordinary) neighborhood of (x_0, y_0) [27, vol 2, p. 305]. There are similar results for circular summability [24].

5. PLESSNER'S THEOREM

Again in one dimension, this is fun, easy to state, and pretty, although somewhat hard to prove. We look at a trigonometric series

$$T = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

and we look at its positive half

$$T^+ = \sum_{n=0}^{\infty} a_n e^{2\pi i n x}.$$

If this general trigonometric series T converges on a set E of positive measure, then T^+ will also converge on the same set up to a set of measure 0. Equivalently, we can look at the trigonometric series conjugate to T

$$\tilde{T} = -\sum_{n=-\infty}^{\infty} (i \operatorname{sgn} n) a_n e^{2\pi i n x}, \qquad \operatorname{sgn} n = \begin{cases} 1, & n > 0 \\ 0, & n = 0 \\ -1, & n < 0 \end{cases}.$$

If we look at T, T^+, and $\tilde{T}$ we can see simple algebraic relationships between them (such as $T + i\tilde{T} = 2T^+$) from which it immediately follows that the connection between T and $\tilde{T}$ is the same as that between T and T^+. So another equivalent statement of Plessner's theorem is that if T converges on E, then $\tilde{T}$ converges almost everywhere on E [27, vol 2, p. 216].

In two variables we start with a double series

$$T = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn} e^{2\pi i(mx+ny)}.$$

Now, what are our analogues of T^+ and $\tilde{T}$ going to be? Well by T^+ we might mean

$$T^{+x} = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} a_{mn} e^{2\pi i(mx+ny)}$$

(this is chopping off half the lattice plane) or we might mean

$$T^{++} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} e^{2\pi i(mx+ny)}$$

(which amounts to just grabbing a quarter of the lattice plane).

For all four methods of convergence, these extensions of Plessner's theorem are false. In fact, we can actually let T be the Fourier series of an $L^1(T^2)$ function and get it to converge on a set E, $|E| > 0$, and get both T^{+x} and T^{++} to diverge almost everywhere on E [2]. (This may clarify my introductory remark about the lack of naturality in grouping the (m, n), $(m, -n)$, $(-m, n)$ and $(-m, -n)$ terms of a series together.)

Again we can salvage something here in a variety of ways. One way is to improve the series to be actually the Fourier series of a function in $L^p(T^2)$ for $p > 1$. Then the theorem will be true for unrestricted rectangular convergence but only in two dimensions. (Not in 3 or more [2].) That's one way—improve T. Or, we can add additional technical hypotheses such as the following: If we assume that T^{+x} is a little bit good *and* that T converges on a set E, then we can make it. (For example, that T^{+x} be summable $(C, 1, 0)$ [2].) Or, another way out of the bag is to use some other notion of conjugate altogether [19].

6. UNIQUENESS

The last topic is the best, or the worst, because here almost all the good theorems are questions. Professor Zygmund's article points out that the problem of uniqueness in one variable is a long way from over because of all the difficulty with sets of uniqueness. Well, in several variables it's even further from being over because we haven't even been able to resolve the higher dimensional analogues of the primitive, simple-minded, first draft version theorem due to Riemann which says that if a trigonometric series converges to 0 at every $x \in T^1$, then actually it isn't there—all its coefficients were 0. That's the beginning of the subject and we can't even do that, in several variables, very often. So

$$\text{if } T_n(x) \to 0 \text{ for every } x, \text{ then all coefficients are } 0, \tag{6.1}$$

is the very pretty one dimensional theorem. It's very simple and nice—we just integrate twice formally and observe that the twice integrated series has second generalized derivative equal to 0, so the twice integrated series is convex, concave, and hence linear, and therefore the original series was 0. This is a very clean argument, but the only known argument. There is no other known way to do it, so far. It would be really nice to have another way of doing this.

In two dimensions we have a couple of positive results; and they're only good for two dimensions. One is

$$\text{If } T_{mn} \to 0 \text{ unrestrictedly rectangularly everywhere, then all } a_{mn} = 0. \tag{6.2}$$

The other one is

$$\text{If } T_r \to 0 \text{ circularly everywhere, then all } a_{mn} = 0. \tag{6.3}$$

Basic to both of these proofs is some work of Victor Shapiro involving uniqueness for spherical Abel summability—a very nice paper [21]. Roger Cooke used his fact about coefficients tending to 0 (that I mentioned earlier) together with Shapiro's theorem (which has an assumption concerning coefficient size) to obtain (6.3) [6]. Theorem (6.2) was obtained *also* using Shapiro's theorem which is really kind of crazy when you think about it—to use something about circular means to end up with a rectangular result [3, p. 422]. Examples 1 and 4 seem to emphasize this point. Nevertheless Theorem (6.2) cannot be done directly by rectangular methods—not by me, anyway. That's the end of the positive results in several variables. Notice I was very careful to say "two dimensions" in those two theorems. So that leaves just about every other question in the field wide open. For example, what if you assume that the series is still two dimensional, but converges only *restrictedly* rectangularly to 0 everywhere—what then? And what if you look at a triple trigonometric series which converges to 0 everywhere and you name the method of convergence—what then?

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For more extensive discussion of the field as a whole and bibliographies, the reader is encouraged to seek out [5], [20], [26], or [14]. The first two of these are mainly about spherical methods with which I deal only lightly. The third has a mammoth bibliography, while the last surveys a lot of topics I do, giving proofs.