

Neither a Worst Convergent Series nor a Best Divergent Series Exists

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When I learned the comparison test for infinite series, the first things I wanted were a “biggest” convergent series and a “smallest” divergent series. Here is a geometric proof of the well-known theorem that guarantees that I am destined to be forever denied both of these two pleasures.

Theorem. *Let $\sum_{n=1}^{\infty} c_n$ be any convergent series with positive terms. Then there exists a convergent series $\sum_{n=1}^{\infty} C_n$ with much bigger terms in the sense that*

$$\lim_{n \rightarrow \infty} \frac{C_n}{c_n} = \infty.$$

Similarly, for any divergent series $\sum_{n=1}^{\infty} D_n$ with positive terms, there exists a divergent series $\sum_{n=2}^{\infty} d_n$ with much smaller terms in the sense that

$$\lim_{n \rightarrow \infty} \frac{d_n}{D_n} = 0.$$

Proof. For each n , let $r_n = c_n + c_{n+1} + \cdots$ and $s_n = D_1 + \cdots + D_n$. Letting

$$C_n = \frac{c_n}{\sqrt{r_n}} \quad \text{and} \quad d_n = \frac{D_n}{s_{n-1}},$$

then $\lim_{n \rightarrow \infty} C_n/c_n = \lim_{n \rightarrow \infty} 1/\sqrt{r_n} = \infty$ and $\lim_{n \rightarrow \infty} d_n/D_n = \lim_{n \rightarrow \infty} 1/s_{n-1} = 0$, so it only remains to check that $\sum C_n$ converges and that $\sum d_n$ diverges. To see that this is indeed the case, simply write $C_n = (1/\sqrt{r_n})(r_n - r_{n+1})$ and $d_n = (1/s_{n-1})(s_n - s_{n-1})$; observe that $\int_0^{r_1} 1/\sqrt{x} \, dx < \infty$ and $\int_{s_1}^{\infty} 1/x \, dx = \infty$; and note that the n th term of series $\sum C_n$ is the area of the gray rectangle in Figure 1a, while the n th term of series $\sum d_n$ is the area of the gray rectangle in Figure 1b. \square

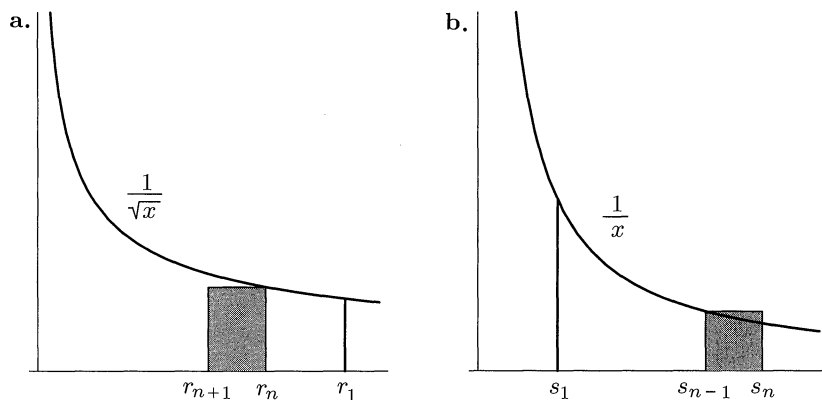


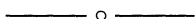
Figure 1

The theorem combines a result of du Bois Reymond [3], who discovered $\sum C_n$, with one of Abel [1], who discovered $\sum d_n$. Generalizations of both examples with

geometric proofs like that above appear in Bary [2]. Bary credits the geometric approach to Denjoy. The theorem appears in several standard textbooks, such as Rudin's *Principles of Mathematical Analysis* and Bartle's *Elements of Real Analysis*.

References

1. N. H. Abel, Note sur le mémoire de Mr. L. Oliver No. 4, du second tome de ce journal, ayant pour titre "remarques sur les séries infinies et leur convergence," *Crelles Journal* 3 (1828) 80. See also: Sur les séries, *Oeuvres complètes*, vol. 2, 2nd ed., Christiania, CITY??, 1881, pp. 197–201.
2. N. Bary, *A Treatise on Trigonometric Series*, Pergamon, London, 1961, pp. 470–471.
3. P. du Bois Reymond, Neue theorie der convergenz und divergenz von reihen mit positiven gliedern, *Crelles Journal* 76 (1873) 61–91.



A Note on Taylor's Series for $\sin(ax + b)$ and $\cos(ax + b)$

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For many students in elementary calculus, finding the Taylor series expansion of $\sin(ax + b)$ and $\cos(ax + b)$ (where a and b are constants with $a \neq 0$) centered at some point $x_0 \neq 0$ is quite formidable. Most calculus textbooks contain few exercises to find such expansions, and these exercises typically ask students to find only the first few terms of these expansions. The purpose of this note is to develop a technique that will allow students to easily determine the complete Taylor expansions.

In elementary calculus, it is shown that

$$\frac{d^n \sin x}{dx^n} = \begin{cases} \sin x, & \text{if } n \equiv 0 \pmod{4} \\ \cos x, & \text{if } n \equiv 1 \pmod{4} \\ -\sin x, & \text{if } n \equiv 2 \pmod{4} \\ -\cos x, & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

Using the addition formulas for the sine and cosine, it is straightforward to show that this can be written simply as

$$\frac{d^n \sin x}{dx^n} = \sin\left(x + \frac{n\pi}{2}\right). \quad (1)$$

Similarly,

$$\frac{d^n \cos x}{dx^n} = \cos\left(x + \frac{n\pi}{2}\right). \quad (2)$$

So, by the chain rule and identities (1) and (2),

$$\frac{d^n \sin(ax + b)}{dx^n} = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

and

$$\frac{d^n \cos(ax + b)}{dx^n} = a^n \cos\left(ax + b + \frac{n\pi}{2}\right).$$

Addition: The last sentence of Reference 1 should be as follows.

See also: Sur les séries, *Oeuvres complètes*, Grøndahl and Son, Christiania(Oslo), 1881, pp. 197–201.