Some Lie algebras generated by reflections

Or, an excuse to learn some classical rep theory

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Underlying question

Question

Let (W, S) be a finite Coxeter group. Consider the group algebra $\mathbb{C}W$ as a Lie algebra, $\mathsf{Lie}(\mathbb{C}W)$, via the commutator bracket

$$[x, y] = xy - yx.$$

What is the structure of the Lie subalgebra generated by S?

Equivalently[1], what is the structure of the Lie subalgebra generated by the set of all reflections (i.e., conjugates of elements of S)?

Motivation

Connections to the braid group in Type A. Other types: curiosity.

[1] To help see the equivalence, expand $[s, [s, x]]$ and use the fact that $s^2 = 1$.

Classical structure theory

First, what is the structure of $\mathsf{Lie}(\mathbb{C}W)$?

Artin–Wedderburn Theorem

Let A be a finite-dimensional associative semisimple algebra over $\mathbb{C}$, and let $V_1, \ldots, V_m$ be a complete set of pairwise non-isomorphic simple A-modules. Then as an associate $\mathbb{C}$-algebra,

$$A \cong \mathsf{End}(V_1) \oplus \cdots \oplus \mathsf{End}(V_m).$$

Corollary

Let W be a finite group, and let $V_1, \ldots, V_m$ be a complete set of pairwise non-isomorphic simple $\mathbb{C}W$-modules. Then

$$\mathsf{Lie}(\mathbb{C}W) \cong \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_m).$$

Reductivity

Let (W, S) be a finite Coxeter group.

Let $V_1, \ldots, V_m$ be a complete set of simple $\mathbb{C}W$-modules.

Let $\mathfrak{s} \subseteq \mathsf{Lie}(\mathbb{C}W)$ be the Lie subalgebra generated by the set $S \subseteq W$.

- Each V_i is a simple $\mathfrak{s}$-module, because $S \subseteq \mathfrak{s}$ and $\langle S \rangle = W$.
- Then $V_1 \oplus \cdots \oplus V_m$ is a faithful, f.d. semisimple $\mathfrak{s}$-module.

Consequences

The Lie algebra $\mathfrak{s}$ is reductive, $\mathfrak{s}' = [\mathfrak{s}, \mathfrak{s}]$ is semisimple, and

$$\mathfrak{s} = \mathfrak{s}' \oplus Z(\mathfrak{s}).$$

The center $Z(\mathfrak{s})$ is spanned by the class sums in $\mathbb{C}W$ of conjugacy classes of elements of S.

Type A: The Lie algebra of Transpositions

I. Marin, L’algèbre de Lie des transpositions, J. Algebra 310 (2007)

The symmetric group

The symmetric group as a Coxeter group

- $\mathfrak{S}_n$ is a Coxeter group with $S = \{(1,2),(2,3),\dots,(n-1,n)\}$.
- The set of all reflections in $\mathfrak{S}_n$ is $\{(i,j) : 1 \leq i < j \leq n\}$.

The simple $\mathbb{C}\mathfrak{S}_n$-modules are labeled by partitions $\lambda \vdash n$.

The group algebra of the symmetric group S_n

For $\lambda \vdash n$, let S^λ be the corresponding simple Specht module. Then

$$\mathbb{C}\mathfrak{S}_n \cong \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda).$$

Thus $\mathrm{Lie}(\mathbb{C}\mathfrak{S}_n) \cong \bigoplus_{\lambda \vdash n} \mathfrak{gl}(\lambda)$, where $\mathfrak{gl}(\lambda) = \mathrm{End}(S^\lambda)$.

Type A: The Lie algebra of transpositions

Let $\mathfrak{s}_n \subseteq \mathsf{Lie}(\mathbb{C}\mathfrak{S}_n)$ be the subalgebra generated by the transpositions.

Then $\mathfrak{s}_n = \mathfrak{s}_n' \oplus Z(\mathfrak{s}_n)$, and $Z(\mathfrak{s}_n)$ is spanned by

$$T_n = \sum_{i<j} (i, j).$$

The Artin–Wedderburn map restricts to a Lie algebra homomorphism

$$\mathfrak{s}_n' \hookrightarrow \bigoplus_{\lambda \vdash n} \mathfrak{sl}(\lambda)$$

What is the image of this map?

Factorizations: Hook partitions

Exterior powers of the reflection representation

The $(n-1)$-dimensional reflection representation of $\mathfrak{S}_n$ is labeled by $\alpha = [n-1, 1]$. Let $\alpha_d = [n-d, 1^d]$. Then for $0 \leq d \leq n-1$,

$$S^{[n-d,1^d]} \cong \Lambda^d(S^\alpha)$$

as $\mathfrak{S}_n$-modules, and also as $\mathfrak{s}'_n$-modules (different coproduct).

Then for $0 \leq d \leq n-1$, the module map $\rho_{\alpha_d} : \mathfrak{s}'_n \to \mathfrak{sl}(\alpha_d)$ factors as

$$\mathfrak{s}'_n \xrightarrow{\rho_\alpha} \mathfrak{sl}(\alpha) \xrightarrow{\Delta_d} \mathfrak{sl}(\alpha_d).$$

Factorizations: Dual partitions

Given $\lambda \vdash n$, let λ^* be the conjugate (transpose) partition.

Group-Theoretic Facts

As $\mathbb{C}\mathfrak{S}_n$-modules, $(S^\lambda)^* \cong S^\lambda$ and $S^\lambda \otimes \mathbf{sgn} \cong S^{\lambda^*}$.

Compare the action of a transposition $s \in \mathfrak{s}_n$ on

- $\mathrm{Res}_{\mathfrak{s}_n}^{\mathbb{C}\mathfrak{S}_n} \left((S^\lambda)^* \otimes \mathbf{sgn}\right)$, restriction of the group-theoretic action
- $(S^\lambda)^{*,\mathbf{Lie}}$, dual space with contragredient Lie algebra action

$$(s.\phi)(v) = -\phi(s^{-1}.v) = -\phi(s.v)$$

Corollary

$S^{\lambda^*} \cong (S^\lambda)^*$ as $\mathfrak{s}_n$-modules, the map $\mathfrak{s}_n \xrightarrow{\rho_\lambda \oplus \rho_{\lambda^*}} \mathfrak{gl}(\lambda) \oplus \mathfrak{gl}(\lambda^*)$ can be written in the form $X \mapsto (X, -X^t)$, and $\mathbf{im}(\rho_\lambda) \cong \mathbf{im}(\rho_{\lambda^*})$.

Factorizations: Self-dual partitions

Suppose $\lambda \vdash n$ and $\lambda = \lambda^*$. Then there exists a linear isomorphism

$$\phi_{\mathsf{sgn}} : S^\lambda \to S^\lambda$$

such that $\phi_{\mathsf{sgn}}(\sigma.v) = \mathsf{sgn}(\sigma)\sigma \cdot \phi_{\mathsf{sgn}}(v)$ for all $\sigma \in \mathfrak{S}_n$ and $v \in S^\lambda$.

Let $\langle -, - \rangle_\lambda$ be a non-degenerate, $\mathfrak{S}_n$-invariant, symmetric bilinear form on S^λ. Define a new bilinear form $(-|-)_\lambda$ on S^λ by

$$(u|v)_\lambda = \langle u, \phi_{\mathsf{sgn}}(v)\rangle_\lambda.$$

Lemma

The form $(-|-)_\lambda$ is either symmetric or anti-symmetric, depending on the sign of the permutation that maps $\lambda \mapsto \lambda^*$, and

$$\rho_\lambda(\mathfrak{s}_n) \subseteq \mathfrak{osp}(\lambda) := \{x \in \mathfrak{gl}(\lambda) : (x.u|v) + (u|x.v) = 0 \ \ \forall\, u, v \in S^\lambda\}.$$

Establish using Young normal form for S^λ to make ϕ_{sgn} and $(-|-)_\lambda$ explicit.

Marin's result in Type A

Let $E_n = \{\lambda \vdash n : \lambda \text{ is not a hook and } \lambda \neq \lambda^*\}$.

Let $F_n = \{\lambda \vdash n : \lambda \text{ is not a hook and } \lambda = \lambda^*\}$.

Let $\sim$ be the relation on $\{\lambda : \lambda \vdash n\}$ generated by $\lambda \sim \lambda^*$.

Theorem [Marin, J. Algebra 310 (2007)]

For $n \geq 2$, the Artin–Wedderburn map induces an isomorphism

$$\mathfrak{s}'_n \cong \mathfrak{sl}(\alpha) \oplus \Big[\bigoplus_{\lambda \in E_n/\sim} \mathfrak{sl}(\lambda) \Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{osp}(\lambda) \Big].$$

Proof is by induction on n.

Ideas behind the proof: show image is simple

Step 1: Compute $\mathfrak{s}_\lambda := \rho_\lambda(\mathfrak{s}'_n) \subset \mathrm{End}(S^\lambda)$.

Consider $\mathfrak{h} := \rho_\lambda(\mathfrak{s}'_{n-1})$ and the multiplicity-free restriction

$$S^\lambda\downarrow_{\mathbb{C}\mathfrak{S}_{n-1}} \cong \bigoplus_{\mu \prec \lambda} S^\mu.$$

Marin, Lemme 15

Let V be a finite-dimensional $\mathbb{C}$-vector space, and let $\mathfrak{h} \subset \mathfrak{g}$ be two semisimple subalgebras of $\mathfrak{sl}(V)$ such that:

1. V is irreducible for the action of $\mathfrak{g}$.
2. The restriction of V to each simple ideal of $\mathfrak{h}$ admits an irreducible factor of multiplicity one.
3. $\mathrm{rk}(\mathfrak{g}) < 2 \cdot \mathrm{rk}(\mathfrak{h})$.

Then $\mathfrak{g}$ is a simple Lie algebra.

Ideas behind the proof: nail down the simple image

Marin, Lemme 13

If $\mathfrak{g}$ is a simple complex Lie algebra that admits an irreducible representation V such that $\dim(V) < 2 \cdot \mathrm{rk}(\mathfrak{g})$, then $\mathfrak{g} \cong \mathfrak{sl}(V)$.

Marin, Lemme 14

Let $(\mathfrak{g}, V)$ be a pair with $\mathfrak{g}$ a simple complex Lie algebra and V a simple $\mathfrak{g}$-module such that

$$2 \cdot \mathrm{rk}(\mathfrak{g}) \leq \dim(V) < 4 \cdot \mathrm{rk}(\mathfrak{g}).$$

Then either

- $\mathfrak{g} \cong \mathfrak{so}(V)$, or
- $\mathfrak{g} \cong \mathfrak{sp}(V)$, or
- $(\mathfrak{g}, V)$ is one of 17 exceptions with $\mathrm{rk}(\mathfrak{g}) \leq 6$ and $\dim(V) \leq 21$.

Ideas behind the proof: surjectivity of Artin–Wedderburn map

Write $\mathfrak{s}_n' = \mathfrak{s}^\lambda \oplus \ker(\rho_\lambda)$, orthogonal decomposition w.r.t. Killing form

Step 2: Show that the simple ideals

$$\{\mathfrak{s}^\alpha\} \cup \{\mathfrak{s}^\lambda : \lambda \in E_n/\sim\} \cup \{\mathfrak{s}^\lambda : \lambda \in F_n\}$$

are distinct. Deduce by dimension comparison that

$$\mathfrak{s}_n' \cong \mathfrak{sl}(\alpha) \oplus \Big[\bigoplus_{\lambda \in E_n/\sim} \mathfrak{sl}(\lambda)\Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{osp}(\lambda)\Big]$$

Marin, Proposition 2

Let V_1 and V_2 be simple $\mathbb{C}\mathfrak{S}_n$-modules of dimension ≥ 2. TFAE:

1. $V_1 \cong V_2$ as $\mathbb{C}\mathfrak{S}_n$-modules
2. $V_1 \cong V_2$ as $\mathfrak{s}_n$-modules
3. $V_1 \cong V_2$ as $\mathfrak{s}_n'$-modules

Type BC: The Hyperoctahedral group

a.k.a. the signed symmetric group

The Hyperoctahedral group

$\mathcal{B}_n \subset GL_n(\mathbb{C})$ is the group of signed permutation matrices.

$\mathcal{B}_n \cong \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ identifies with set of permutation matrices.

Let t_i be the multiplicative generator for i-th factor of $\mathbb{Z}_2$, i.e., the diagonal matrix whose i-th entry is -1 and other entries are $+1$.

Coxeter generators are $\{s_1, \ldots, s_{n-1}, t_n\}$. The set of reflections is

$$\big\{(i,j), t_i t_j (i,j) : 1 \leq i < j \leq n\big\} \cup \{t_i : 1 \leq i \leq n\}$$

Three nontrivial linear characters ε, ε', and ε''

$$\varepsilon(t_i) = -1, \qquad \varepsilon'(t_i) = -1, \qquad \varepsilon''(t_i) = +1,$$
$$\varepsilon(s_j) = -1, \qquad \varepsilon'(s_j) = +1, \qquad \varepsilon''(s_j) = -1.$$

ε is the sign character of $\mathcal{B}_n$.

Simple modules in Type BC

Simple modules labeled by bipartitions $(\lambda, \mu) \vdash n$, i.e., ordered pairs of partitions λ and μ such that $|\lambda| + |\mu| = n$.

Considering S^λ as a $\mathcal{B}_a$-module via inflation along the quotient map $\mathcal{B}_a \twoheadrightarrow \mathfrak{S}_a$ (and similarly for S^μ), one has

$$S^{(\lambda,\mu)} = \mathrm{Ind}_{\mathcal{B}_a \times \mathcal{B}_b}^{\mathcal{B}_n} \left(S^\lambda \boxtimes (S^\mu \otimes \varepsilon') \right).$$

Effect of tensor product with linear characters

$$S^{(\lambda,\mu)} \otimes \varepsilon \cong S^{(\mu^*,\lambda^*)}, \quad S^{(\lambda,\mu)} \otimes \varepsilon' \cong S^{(\mu,\lambda)}, \quad S^{(\lambda,\mu)} \otimes \varepsilon'' \cong S^{(\lambda^*,\mu^*)}.$$

One-dimensional modules

$$S^{([n],\emptyset)} = \mathbb{C}, \quad S^{([1^n],\emptyset)} = \varepsilon'', \quad S^{(\emptyset,[n])} = \varepsilon', \quad S^{(\emptyset,[1^n])} = \varepsilon.$$

Type B: The Lie algebra of reflections

Let $\mathfrak{b}_n \subseteq \mathbb{C}\mathcal{B}_n$ be the Lie algebra generated by the reflections in $\mathcal{B}_n$.

Then $\mathfrak{b}_n = \mathfrak{b}_n' \oplus Z(\mathfrak{b}_n)$, and $Z(\mathfrak{b}_n)$ is spanned by

$$\mathcal{X}_n = \sum_{i<j} \left[(i,j) + t_i t_j (i,j)\right] \quad \text{and} \quad \mathcal{T}_n = \sum_{i=1}^{n} t_i.$$

The Artin–Wedderburn isomorphism restricts to a Lie algebra map

$$\mathfrak{b}_n' \to \bigoplus_{(\lambda,\mu)\in\mathcal{BP}(n)} \mathfrak{sl}(\lambda,\mu).$$

Set $\mathfrak{b}_{(\lambda,\mu)} = \rho_{(\lambda,\mu)}(\mathfrak{b}_n') \subseteq \mathfrak{sl}(\lambda,\mu) \subset \mathrm{End}(S^{(\lambda,\mu)})$.

Factorizations arising from dual pairs

Dual pairs (λ, μ) and (μ^*, λ^*)

$[S^{(\lambda,\mu)}]^* \cong S^{(\mu^*,\lambda^*)}$ as $\mathfrak{b}_n$-modules.

Then $\rho_{(\mu^*,\lambda^*)}$ factors through $\rho_{(\lambda,\mu)}$, and $\mathfrak{b}_{(\lambda,\mu)} \cong \mathfrak{b}_{(\mu^*,\lambda^*)}$.

Self-dual pairs (λ, λ^*)

If $\lambda \vdash n/2$, then there is a non-degenerate bilinear form $(-|-)_{(\lambda,\lambda^*)}$ on $S^{(\lambda,\lambda^*)}$, that is either symmetric or anti-symmetric depending on the parity of $n/2$, and such that

$$\begin{aligned}\mathfrak{b}_{(\lambda,\lambda^*)} &\subseteq \mathfrak{osp}(\lambda,\lambda^*)\\ &:= \{x \in \mathfrak{gl}(\lambda,\lambda^*) : (x.u|v) + (u|x.v) = 0 \ \ \forall u, v \in S^{(\lambda,\lambda^*)}\}.\end{aligned}$$

Establish with help of normal form on simple modules described by Mishra and Srinivasan in *The Okounkov–Vershik approach to the representation theory of $G \sim S_n$*, J. Algebr. Comb. 44 (2016).

Factorizations: Arm and leg bipartitions

The natural module of $\mathcal{B}_n \subset GL_n(\mathbb{C})$ is labeled by $\beta = ([n-1],[1])$.

Let $\gamma = ([1],[n-1])$.

Exterior powers for arm and leg bipartitions

For $0 < d < n$, set $\beta_d = ([n-d],[1^d])$ and $\gamma_d = ([1^d],[n-d])$. Then

$$S^{\beta_d} \cong \Lambda^d(S^\beta) \quad \text{and} \quad S^{\gamma_d} \cong \Lambda^d(S^\gamma)$$

as $\mathfrak{b}'_n$-modules. Then the module maps ρ_{β_d} and ρ_{γ_d} factor as

$$\mathfrak{b}'_n \xrightarrow{\rho_\beta} \mathfrak{sl}(\beta) \xrightarrow{\Delta_d} \mathfrak{sl}(\beta_d) \quad \text{and} \quad \mathfrak{b}'_n \xrightarrow{\rho_\gamma} \mathfrak{sl}(\beta) \xrightarrow{\Delta_d} \mathfrak{sl}(\gamma_d).$$

Then $\mathfrak{b}_\beta \cong \mathfrak{b}_{\beta_d}$ and $\mathfrak{b}_\gamma \cong \mathfrak{b}_{\gamma_d}$ for $0 < d < n$.

Factorizations: Improper bipartitions

The split sequence of groups $\mathfrak{S}_n \hookrightarrow \mathcal{B}_n \twoheadrightarrow \mathfrak{S}_n$ restricts to

$$\mathfrak{s}'_n \hookrightarrow \mathfrak{b}'_n \twoheadrightarrow \mathfrak{s}'_n.$$

As modules over the subalgebra $\mathbb{C}\mathfrak{S}_n \subset \mathbb{C}\mathcal{B}_n$,

$$S^{(\lambda,\emptyset)} = S^{\lambda} = S^{(\emptyset,\lambda)}.$$

Improper bipartitions

Let $\lambda \vdash n$. Then the maps $\mathfrak{s}'_n \hookrightarrow \mathfrak{b}'_n \twoheadrightarrow \mathfrak{s}'_n$ induce equalities

$$\mathfrak{b}_{(\lambda,\emptyset)} = \mathfrak{s}_{\lambda} = \mathfrak{b}_{(\emptyset,\lambda)}.$$

Difference from type A: $\mathfrak{b}'_n$ can distinguish *most* simple $\mathbb{C}\mathcal{B}_n$-modules, but not pairs of the form $S^{(\lambda,\emptyset)}$ and $S^{(\emptyset,\lambda)}$.

Main result in Type BC

Let $E(n) = \{(\lambda, \mu) \vdash n : (\lambda, \mu)$ is proper, not an A&L, and $(\lambda, \mu) \neq (\mu^*, \lambda^*)\}$.

Let $F(n) = \{(\lambda, \lambda^*) \vdash n : (\lambda, \mu)$ is proper and not an A&L$\}$.

Let $\sim$ be the relation generated by $(\lambda, \mu) \sim (\mu^*, \lambda^*)$.

Theorem (Drupieski and Kujawa, 2025)

For $n \geq 2$, the Artin–Wedderburn map induces an isomorphism

$$\mathfrak{b}'_n \cong \mathfrak{s}'_n \oplus \mathfrak{sl}(\beta) \oplus \mathfrak{sl}(\gamma) \oplus \left[\bigoplus_{(\lambda,\mu) \in E(n)/\sim} \mathfrak{sl}(\lambda, \mu) \right] \oplus \left[\bigoplus_{(\lambda,\mu) \in F(n)} \mathfrak{osp}(\lambda, \mu) \right].$$

Proof by induction on n, using the restriction formula

$$S^{(\lambda,\mu)}\downarrow_{\mathbb{C}\mathcal{B}_{n-1}} \cong \left[\bigoplus_{\nu \prec \lambda} S^{(\nu,\lambda)} \right] \oplus \left[\bigoplus_{\tau \prec \mu} S^{(\lambda,\tau)} \right]$$

Type D: The Demihyperoctahedral group

(or maybe the group of even-signed permutations?)

The Demihyperoctahedral group

$\mathcal{D}_n \subset \mathcal{B}_n$ is the kernel of the character $\varepsilon' : \mathcal{B}_n \to \{\pm 1\}$ defined by

$$\varepsilon'(t_i) = -1 \quad \text{and} \quad \varepsilon'(s_j) = 1.$$

Coxeter generators are $\left\{s_1, \ldots, s_{n-1}, \widetilde{s}_n\right\}$, where $\widetilde{s}_n = t_{n-1} t_n s_{n-1}$.

The set of reflections in $\mathcal{D}_n$ is

$$\left\{(i,j), t_i t_j (i,j) : 1 \leq i < j \leq n\right\}.$$

The sign character of $\mathcal{D}_n$ is the restriction of the linear character ε'' on $\mathcal{B}_n$, which was induced by the sign character on $\mathfrak{S}_n$.

Simple modules in Type D

$\mathcal{D}_n = \ker(\varepsilon')$. Recall that $S^{(\lambda,\mu)} \otimes \varepsilon' \cong S^{(\mu,\lambda)}$.

Classification of simple $\mathbb{C}\mathcal{D}_n$-modules

Up to isomorphism, each simple $\mathbb{C}\mathcal{D}_n$-module arises via either:

1. If $(\lambda,\mu) \vdash n$ and $\lambda \neq \mu$, then $\mathrm{Res}^{\mathcal{B}_n}_{\mathcal{D}_n}(S^{(\lambda,\mu)}) \cong \mathrm{Res}^{\mathcal{B}_n}_{\mathcal{D}_n}(S^{(\mu,\lambda)})$ is a simple module, denoted $S^{\{\lambda,\mu\}}$.
2. If $(\lambda,\lambda) \vdash n$, then $\mathrm{Res}^{\mathcal{B}_n}_{\mathcal{D}_n}(S^{(\lambda,\lambda)})$ is the sum of two non-isomorphic simple $\mathbb{C}\mathcal{D}_n$-modules $S^{\{\lambda,+\}}$ and $S^{\{\lambda,-\}}$.

Restriction formulas:

$$S^{\{\lambda,\mu\}}\downarrow_{\mathbb{C}\mathcal{D}_{n-1}} \cong \left[\bigoplus_{\nu \prec \lambda} S^{\{\nu,\lambda\}}\right] \oplus \left[\bigoplus_{\tau \prec \mu} S^{\{\lambda,\tau\}}\right],$$

$$S^{\{\lambda,\pm\}}\downarrow_{\mathbb{C}\mathcal{D}_{n-1}} \cong \bigoplus_{\nu \prec \lambda} S^{\{\nu,\lambda\}}$$

Tensor products with the sign character

If $(\lambda, \mu) \vdash n$ and $\lambda \neq \mu$, then

$$S^{\{\lambda,\mu\}} \otimes \varepsilon'' \cong S^{\{\lambda^*,\mu^*\}}.$$

If $(\lambda, \lambda) \vdash n$, then

$$S^{\{\lambda,\pm\}} \otimes \varepsilon'' \cong \begin{cases} S^{\{\lambda^*,\pm\}} & \text{if } n/2 \text{ is even,} \\ S^{\{\lambda^*,\mp\}} & \text{if } n/2 \text{ is odd;} \end{cases}$$

Main result in Type D

Let $E\{n\} = \{\{\lambda, \mu\} \vdash n : \{\lambda, \mu\}$ is proper, not an A&L, and $\{\lambda, \mu\} \neq \{\lambda^*, \mu^*\}\}$.

Let $F\{n\} = \{\{\lambda, \mu\} \vdash n : \{\lambda, \mu\}$ is proper, not an A&L, and $\{\lambda, \mu\} = \{\lambda^*, \mu^*\}\}$.

Let $\sim$ be the relation generated by $\{\lambda, \mu\} \sim \{\lambda^*, \mu^*\}$.

Theorem (Drupieski and Kujawa, 2025)

$$\begin{aligned}\mathfrak{d}_n' \cong \mathfrak{s}_n' \oplus \mathfrak{sl}(\beta) & \\ &\oplus \Big[\bigoplus_{\substack{\{\lambda,\mu\} \in E\{n\}/\sim \\ \lambda \neq \mu}} \mathfrak{sl}\{\lambda, \mu\} \Big] \oplus \Big[\bigoplus_{\{\lambda,\lambda\} \in E\{n\}/\sim} \mathfrak{sl}\{\lambda, +\} \oplus \mathfrak{sl}\{\lambda, -\} \Big] \\ &\oplus \Big[\bigoplus_{\substack{\{\lambda,\mu\} \in F\{n\} \\ \lambda \neq \mu}} \mathfrak{osp}\{\lambda, \mu\} \Big] \oplus \Big[\bigoplus_{\{\lambda,\lambda\} \in F\{n\}} \mathfrak{d}_{\{\lambda,+\}} \oplus \mathfrak{d}^{\star}_{\{\lambda,-\}} \Big],\end{aligned}$$

The unspecified terms $\mathfrak{d}_{\{\lambda,\pm\}}$ are either $\mathfrak{osp}\{\lambda, \pm\}$ or $\mathfrak{sl}\{\lambda, \pm\}$ depending the parity of $n/2$, and the $\star$ means $\mathfrak{d}_{\{\lambda,-\}}$ is omitted entirely if $n/2$ is odd.

Difference from types ABC: Rank considerations alone seem insufficient to show $\mathfrak{d}_{\{\lambda,+\}}$ is all of $\mathfrak{sl}\{\lambda, +\}$ and not just $\mathfrak{osp}\{\lambda, +\}$ if $n/2$ is odd.

Types G_2, F_4, and beyond

here be dragons

Types G and F

The Lie algebra generated by the reflections in the Weyl group $W(G_2)$ is 8-dimensional, with 2-dimensional center. Deduce that the rest is two copies of $\mathfrak{sl}(2)$.

The Lie algebra generated by the reflections in the Weyl group $W(F_4)$ is 510-dimensional.

GAP Calculations in Type F

Character	Dimension	Dual	image
X1	1	X2	0
X3	1	X4	0
X5	2	X6	3
X7	2	X8	3
X9	4	X12	15
X10	4	X11	15
X13	4	X13	6
X14	6	X14	15
X15	6	X15	15
X16	8	X17	63
X18	8	X19	63
X20	9	X23	80
X21	9	X22	80
X24	12	X24	66
X25	16	X25	120
		Total	544
		Target	508
		Defecit	-36

The Lie superalgebra of transpositions

Drupieski and Kujawa, arXiv:2310.01555

The group algebra of the symmetric group, as a superalgebra

The symmetric group $\mathfrak{S}_n$ is a supergroup, with

- $(\mathfrak{S}_n)_{\bar{0}} = A_n$, the alternating group.
- $(\mathfrak{S}_n)_{\bar{1}} = \mathfrak{S}_n \backslash A_n$, the set of odd permutations.

This extends to a $\mathbb{Z}_2$-grading on the group algebra $\mathbb{C}\mathfrak{S}_n$, with

- $(\mathbb{C}\mathfrak{S}_n)_{\bar{0}} = \mathbb{C}A_n$, the group algebra of the alternating group.

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Consider the group algebra of the symmetric group $\mathbb{C}S_n$. Then there is a corresponding Lie algebra $\mathfrak{L}(S_n)$ defined by

$$[\sigma, \tau] = \sigma \circ \tau - \tau \circ \sigma,$$

where $\sigma, \tau \in S_n$. The structure of $\mathfrak{L}(S_n)$ in terms of simple factors has been considered in this post. One can also ask the same question for the Lie subalgebra of $\mathfrak{L}(S_n)$ generated by transpositions, which was considered in this post.

Now, since there is a $\mathbb{Z}_2$ grading of $\mathbb{C}S_n$, one can also define a Lie superalgebra $s\mathfrak{L}(S_n)$ on it by replacing the commutators with anti-commutators

$$\{\sigma, \tau\} = \sigma \circ \tau + \tau \circ \sigma,$$

for all $\sigma, \tau \in S_n^{(1)}$, where $S_n^{(1)}$ is the odd part of the symmetric group, and all other commutators remain unchanged. Now we have similar questions: what is the structure of $s\mathfrak{L}(S_n)$ in terms of simple Lie superalgebras? What is the subalgebra of $s\mathfrak{L}(S_n)$ generated by transpositions?

My attempt is for $n = 3$, $s\mathfrak{L}(S_n) \cong \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$, while the subalgebra generated by transpositions is $\mathfrak{sl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$. I think in general $s\mathfrak{L}(S_n)$ should be very similar to $\mathfrak{L}(S_n)$, but it might be much harder to determine the subalgebra generated by transpositions.

rt.representation-theory lie-algebras permutations symmetric-groups lie-superalgebras

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edited Oct 4 at 16:50
Jules Lamers

asked Aug 8, 2022 at 20:26
WunderNatur

Lie superalgebra generated by transpositions

Let $\mathfrak{g}_n \subset \mathbb{C}\mathfrak{S}_n$ be the Lie superalgebra generated by all transpositions.

Theorem (Drupieski and Kujawa, 2024)

$$\mathfrak{g}_n = \mathfrak{D}(\mathbb{C}\mathfrak{S}_n) \oplus \mathbb{C} \cdot T_n,$$

where

$$\mathfrak{D}(\mathbb{C}\mathfrak{S}_n) \cong \Big[\bigoplus_{\{\lambda \vdash n : \lambda \neq \lambda^*\}/\sim} \mathfrak{sq}(W^\lambda) \Big] \oplus \Big[\bigoplus_{\{\lambda \vdash n : \lambda = \lambda^*\}} \mathfrak{sl}(W^\lambda) \Big]$$

$$\mathfrak{sq}(W^\lambda) \cong \left\{ \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right] \in \mathfrak{gl}(m|m) : \mathbf{tr}(B) = 0 \right\} \qquad m = \mathbf{dim}(S^\lambda),$$

$$\mathfrak{sl}(W^\lambda) \cong \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathfrak{gl}(m|m) : \mathbf{tr}(A) - \mathbf{tr}(D) = 0 \right\} \quad m = \tfrac{1}{2}\,\mathbf{dim}(S^\lambda).$$