Lie (super)algebras generated by reflections

in finite reflection groups

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Underlying question

Question

Let W be a finite (irreducible, real) reflection group. Consider $\mathbb{C}W$ as a Lie algebra, $\mathsf{Lie}(\mathbb{C}W)$, via the commutator bracket

$$[x, y] = xy - yx.$$

What is the structure of the Lie subalgebra $\mathfrak{s}$ generated by the set S of reflections in W?

Motivations

- Connections to the braid group in Type A (Marin, 2000s).
- A question on MathOverflow (Wundernatur, 2022).

Artin–Wedderburn Decomposition

Let $V_1, \ldots, V_m$ be a complete set of simple $\mathbb{C}W$-modules. Then

$$\mathbb{C}W \cong \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_m)$$

as an associative algebra, and hence

$$\mathrm{Lie}(\mathbb{C}W) \cong \mathfrak{gl}(V_1) \oplus \cdots \oplus \mathfrak{gl}(V_m).$$

Study $\mathfrak{s}$ by way of its image under this isomorphism.

Reductivity

- Each V_i is a simple $\mathfrak{s}$-module, because $S \subseteq \mathfrak{s}$ and $\langle S \rangle = W$.
- Then $V_1 \oplus \cdots \oplus V_m$ is a faithful, f.d. semisimple $\mathfrak{s}$-module.

Consequences

The Lie algebra $\mathfrak{s}$ is reductive, $\mathfrak{s}' = [\mathfrak{s}, \mathfrak{s}]$ is semisimple, and

$$\mathfrak{s} = \mathfrak{s}' \oplus Z(\mathfrak{s}).$$

$Z(\mathfrak{s})$ is spanned by class sums of conjugacy classes of reflections.

Type A: The Symmetric Group

I. Marin, L’algèbre de Lie des transpositions, J. Algebra 310 (2007)

Type A: The Lie algebra of transpositions

Let $\mathfrak{s}_n \subseteq \mathsf{Lie}(\mathbb{C}\mathfrak{S}_n)$ be the subalgebra generated by the transpositions.

Then $\mathfrak{s}_n = \mathfrak{s}'_n \oplus Z(\mathfrak{s}_n)$, and $Z(\mathfrak{s}_n)$ is spanned by $T_n = \sum_{i<j}(i,j)$.

The group algebra of the symmetric group S_n

For $\lambda \vdash n$, let S^λ be the corresponding simple Specht module. Then

$$\mathbb{C}\mathfrak{S}_n \cong \bigoplus_{\lambda \vdash n} \mathsf{End}(S^\lambda).$$

Thus $\mathsf{Lie}(\mathbb{C}\mathfrak{S}_n) \cong \bigoplus_{\lambda \vdash n} \mathfrak{gl}(\lambda)$, where $\mathfrak{gl}(\lambda) = \mathsf{End}(S^\lambda)$.

Then $\mathfrak{s}'_n \hookrightarrow \bigoplus_{\lambda \vdash n} \mathfrak{sl}(S^\lambda)$, but in general the image is much smaller.

Factorizations: Hook partitions

Exterior powers of the reflection representation

The $(n-1)$-dimensional reflection representation of $\mathfrak{S}_n$ is labeled by $\alpha = [n-1, 1]$. Then for $0 \leq d \leq n$,

$$S^{[n-d,1^d]} \cong \Lambda^d(S^\alpha)$$

as $\mathbb{C}\mathfrak{S}_n$-modules, and also as $\mathfrak{s}'_n$-modules (different coproduct).

Let $\alpha_d = [n-d, 1^d]$ (a hook partition).

Then for $0 \leq d \leq n-1$, the module map $\rho_{\alpha_d} : \mathfrak{s}'_n \to \mathfrak{sl}(\alpha_d)$ factors as

$$\mathfrak{s}'_n \xrightarrow{\rho_\alpha} \mathfrak{sl}(\alpha) \xrightarrow{\Delta_d} \mathfrak{sl}(\alpha_d).$$

Factorizations: Dual partitions

Given $\lambda \vdash n$, let λ^* be the conjugate (transpose) partition.

As $\mathbb{C}\mathfrak{S}_n$-modules, $(S^\lambda)^* \cong S^\lambda$ and $S^\lambda \otimes \mathbf{sgn} \cong S^{\lambda^*}$.

Compare the action of a transposition s on

- $(S^\lambda)^* \otimes \mathbf{sgn}$ (restriction of the group-theoretic action)
- $(S^\lambda)^{*,\mathbf{Lie}}$ (dual space with contragredient Lie algebra action)

$$(s.\phi)(v) = -\phi(s^{-1}.v) = -\phi(s.v)$$

Corollary

$S^{\lambda^*} \cong (S^\lambda)^*$ as $\mathfrak{s}_n$-modules. For $\lambda \neq \lambda^*$, the map

$$\mathfrak{s}_n \xrightarrow{\rho_\lambda \oplus \rho_{\lambda^*}} \mathfrak{gl}(\lambda) \oplus \mathfrak{gl}(\lambda^*)$$

can be written in the form $X \mapsto (X, -X^t)$, and $\mathbf{im}(\rho_\lambda) \cong \mathbf{im}(\rho_{\lambda^*})$.

Factorizations: Self-dual partitions

Suppose $S^\lambda \otimes \mathsf{sgn} \cong S^\lambda$, i.e., $\lambda = \lambda^*$. Then there exists a linear iso.

$$\phi_{\mathsf{sgn}} : S^\lambda \to S^\lambda$$

such that $\phi_{\mathsf{sgn}}(\sigma.v) = \mathsf{sgn}(\sigma)\sigma \cdot \phi_{\mathsf{sgn}}(v)$ for all $\sigma \in \mathfrak{S}_n$ and $v \in S^\lambda$.

Let $\langle -, - \rangle_\lambda$ be a non-degenerate, $\mathfrak{S}_n$-invariant, symmetric bilinear form on S^λ. Define a new bilinear form $(-|-)_\lambda$ on S^λ by

$$(u|v)_\lambda = \langle u, \phi_{\mathsf{sgn}}(v) \rangle_\lambda .$$

Lemma

The form $(-|-)_\lambda$ is either symmetric or anti-symmetric, depending on the sign of the permutation that maps $\lambda \mapsto \lambda^*$, and

$$\rho_\lambda(\mathfrak{s}_n) \subseteq \mathfrak{osp}(\lambda) := \{x \in \mathfrak{gl}(\lambda) : (x.u|v) + (u|x.v) = 0 \ \ \forall u, v \in S^\lambda\}.$$

Marin's result in Type A

Let $E_n = \{\lambda \vdash n : \lambda \text{ is not a hook and } \lambda \neq \lambda^*\}$.

Let $F_n = \{\lambda \vdash n : \lambda \text{ is not a hook and } \lambda = \lambda^*\}$.

Let $\sim$ be the relation on $\{\lambda : \lambda \vdash n\}$ generated by $\lambda \sim \lambda^*$.

Theorem [Marin, J. Algebra 310 (2007)]

For $n \geq 2$, the Artin–Wedderburn map induces an isomorphism

$$\mathfrak{s}'_n \cong \mathfrak{sl}(\alpha) \oplus \Big[\bigoplus_{\lambda \in E_n/\sim} \mathfrak{sl}(\lambda) \Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{osp}(\lambda) \Big].$$

Ideas behind the proof

Proof is by induction on n.

Step 1: Compute $\mathfrak{s}_\lambda := \rho_\lambda(\mathfrak{s}'_n) \subset \mathrm{End}(S^\lambda)$.

Consider $\mathfrak{h}_\lambda := \rho_\lambda(\mathfrak{s}'_{n-1})$ and the restriction $S^\lambda\downarrow_{\mathbb{C}\mathfrak{S}_{n-1}} \cong \bigoplus_{\mu \prec \lambda} S^\mu$.

- If S^λ is good[1] for $\mathfrak{h}_\lambda$, then one can deduce that $\mathfrak{s}_\lambda$ is simple.
- Then if $\mathrm{dim}(S^\lambda) < 4 \cdot \mathrm{rk}(\mathfrak{s}_\lambda)$, and if $\mathrm{rk}(\mathfrak{s}_\lambda)$ and $\mathrm{dim}(S^\lambda)$ are not too small[2], then either $\mathfrak{s}_\lambda \cong \mathfrak{sl}(S^\lambda)$ or $\mathfrak{s}_\lambda \cong \mathfrak{osp}(S^\lambda)$.

[1] e.g., if S^λ is multiplicity-free for $\mathfrak{h}_\lambda$ and if $\mathrm{rk}(\mathfrak{s}_\lambda) < 2 \cdot \mathrm{rk}(\mathfrak{h}_\lambda)$.
[2] There are 17 exceptional cases in ranks ≤ 6 and dimensions ≤ 21.

Ideas behind the proof

Write $\mathfrak{s}_n' = \mathfrak{s}^\lambda \oplus \ker(\rho_\lambda)$, orthogonal decomposition with respect to the Killing form. So $\mathfrak{s}^\lambda \cong \mathfrak{s}_\lambda$.

Step 2: Show that

$$\{\mathfrak{s}^\alpha\} \cup \{\mathfrak{s}^\lambda : \lambda \in E_n/\sim\} \cup \{\mathfrak{s}^\lambda : \lambda \in F_n\}$$

are distinct simple ideals in $\mathfrak{s}_n'$, using the fact that $\mathfrak{s}_n'$ can distinguish the simple $\mathbb{C}\mathfrak{S}_n$-modules.[3]

Deduce by dimension comparison that

$$\mathfrak{s}_n' \cong \mathfrak{sl}(\alpha) \oplus \Big[\bigoplus_{\lambda \in E_n/\sim} \mathfrak{sl}(\lambda)\Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{osp}(\lambda)\Big].$$

[3] If W has two root lengths, then things get more complicated.

Types BC and D

[DK, arXiv:2506.01198]

The Hyperoctahedral group

$\mathcal{B}_n \subset GL_n(\mathbb{C})$ is the group of signed permutation matrices.

$\mathcal{B}_n \cong \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ identifies with set of permutation matrices.

Simple modules in Type B

Simple $\mathbb{C}\mathcal{B}_n$-modules are labeled by bipartitions

$$(\lambda, \mu) \vdash n,$$

i.e., ordered pairs of partitions λ and μ such that $|\lambda| + |\mu| = n$.

Call (λ, μ) **improper** if $\lambda = \emptyset$ or $\mu = \emptyset$, and **proper** otherwise.

Call (λ, μ) an **arm-and-leg** (A&L) if it is of the form

$$([i], [1^{n-i}]) \quad \text{or} \quad ([1^i], [n-i])$$

Factorizations: Improper bipartitions

As modules via the quotient $\mathbb{C}\mathcal{B}_n \twoheadrightarrow \mathbb{C}\mathfrak{S}_n$,

$$S^{(\lambda,\emptyset)} = S^{\lambda} = S^{(\emptyset,\lambda)}.$$

Let $\mathfrak{b}_n \subseteq \mathbb{C}\mathcal{B}_n$ be the Lie algebra generated by the reflections in $\mathcal{B}_n$.

The split sequence of groups $\mathfrak{S}_n \hookrightarrow \mathcal{B}_n \twoheadrightarrow \mathfrak{S}_n$ restricts to

$$\mathfrak{s}'_n \hookrightarrow \mathfrak{b}'_n \twoheadrightarrow \mathfrak{s}'_n.$$

Improper bipartitions

Let $\lambda \vdash n$. Then the maps $\mathfrak{s}'_n \hookrightarrow \mathfrak{b}'_n \twoheadrightarrow \mathfrak{s}'_n$ induce equalities

$$\mathfrak{b}_{(\lambda,\emptyset)} = \mathfrak{s}_{\lambda} = \mathfrak{b}_{(\emptyset,\lambda)}.$$

Main result in Types BC and D

Let $E(n) = \{(\lambda, \mu) \vdash n : (\lambda, \mu) \text{ is proper, not an A\&L, and } (\lambda, \mu) \neq (\mu^*, \lambda^*)\}$.

Let $F(n) = \{(\lambda, \lambda^*) \vdash n : (\lambda, \mu) \text{ is proper and not an A\&L}\}$.

Let $\sim$ be the relation generated by $(\lambda, \mu) \sim (\mu^*, \lambda^*)$.

Let $\beta = ([n-1], [1])$ and let $\gamma = ([1], [n-1])$.

Theorem (Drupieski and Kujawa, 2025)

For $n \geq 2$, the Artin–Wedderburn map induces an isomorphism

$$\mathfrak{b}'_n \cong \mathfrak{s}'_n \oplus \mathfrak{sl}(\beta) \oplus \mathfrak{sl}(\gamma) \oplus \Big[\bigoplus_{(\lambda,\mu) \in E(n)/\sim} \mathfrak{sl}(\lambda, \mu) \Big] \oplus \Big[\bigoplus_{(\lambda,\mu) \in F(n)} \mathfrak{osp}(\lambda, \mu) \Big].$$

An analogous (but more complicated to state) result also holds in Type D.

Non-Classical Types

Type $I_2(m)$ for $m \geq 5$, and type H_3

Type $I_2(m)$, i.e., dihedral group D_{2m} or order m

The Lie algebra generated by the reflections in D_{2n} is

$$\begin{cases} \mathfrak{sl}(2)^{\times k} \times \mathbb{C} & \text{if } m = 2k+1, \\ \mathfrak{sl}(2)^{\times (k-1)} \times \mathbb{C} \times \mathbb{C} & \text{if } m = 2k. \end{cases}$$

In particular, the Lie algebra generated by the reflections in $W(G_2)$ is

$$\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathbb{C} \times \mathbb{C}.$$

Type H_3 (icosahedral group)

By dimension comparison via GAP,

$$\mathfrak{s} \cong \mathbb{C} \times \mathfrak{sl}(3) \times \mathfrak{sl}(3) \times \mathfrak{sl}(4) \times \mathfrak{sl}(5)$$

GAP Calculations in Type F_4

Character	Dual	Dimension	sl(V)	so(V)	sp(V)	GAP		
X1	X2	1	0	0	0	0	0	
X3	X4	1	0	0	0	0	0	
X5	X6	2	3	3	3	3	3	
X7	X8	2	3	3	3	3	3	
X9	X12	4	15	15	15	15	15	
X10	X11	4	15	15	15	15	15	
X13	X13	4	15	6	10	6		redundant of X5, X7 per GAP
X14	X14	6	35			15		
X15	X15	6	35	15	21	15		redundant of X10 per GAP
X16	X17	8	63	63	63	63	63	
X18	X19	8	63	63	63	63	63	
X20	X23	9	80	80	80	80	80	
X21	X22	9	80	80	80	80	80	
X24	X24	12	143	66	78	66	66	
X25	X25	16	255	120	136	120	120	
			805	529	567	544	508	Total
			508	508	508	508	508	Lie algebra target
			297	21	59	36	0	Excess

Representation of S3 x S3 inflated to W(F4)

Induced from 3-dimensional reps of W(B4)

$$\mathfrak{s} \cong \mathbb{C} \times \mathbb{C} \times \mathfrak{sl}(2)^{\times 2} \times \mathfrak{so}(4)^{\times 2} \times \mathfrak{sl}(8)^{\times 2} \times \mathfrak{sl}(9) \times \mathfrak{so}(12) \times \mathfrak{so}(16)$$

Calculations for $W(H_4)$

Character	Dual	Dimension	sl(V)	so(V)		sp(V)	
X1	X2	1	0	0	0	0	
X3	X5	4	15	15	15	15	
X4	X6	4	15	15	15	15	
X7	X7	6	35				
X8	X8	6	35				
X9	X9	8	63	28	28	36	
X10	X10	8	63	28		36	X9-X10 blocks together are 28 dim
X11	X13	9	80	80	80	80	
X12	X14	9	80	80	80	80	
X15	X15	10	99	45	45	55	
X16	X17	16	255	255	255	255	
X18	X19	16	255	255	255	255	
X20	X20	16	255	120	120	136	
X21	X21	16	255	120	120	136	
X22	X22	18	323	153	153	171	
X23	X23	24	575	276	276	300	
X24	X24	24	575	276	276	300	
X25	X25	24	575	276	276	300	
X26	X26	24	575	276	276	300	
X27	X28	25	624	624	624	624	
X29	X29	30	899	435	435	465	
X30	X30	30	899	435	435	465	
X31	X32	36	1295	1295	1295	1295	
X33	X33	40	1599	780	780	820	
X34	X34	48	2303	1128	1128	1176	
			11747	6995	6967	7315	Sum
			6967	6967	6967	6967	Lie algebra target
			4780	28	0	348	Overage

Calculations for E6, E7, E8

Restricting from E_6 to D_5:

	D5	Character	1	2	3	4	5	6	7	8	9	10	12	11	13	14	15	16	17	18			
		Dual	2	1	4	3	6	5	8	7	9	12	10	13	11	14	16	15	18	17			
		Dimension	1	1	4	4	5	5	5	5	6	10	10	10	10	10	15	15	20	20			
		LieAlg Rank	0	0	3	3	4	4	4	4	3	9	9	4	4	5	14	14	19	19			
E6																							
Character	Dual	Dimension																			rk(h)	2*rk(h)-dim(V)	
1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	
2	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	
3	4	6	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	4	2	
4	3	6	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	4	2	4*rk(h)-dim(V)
5	5	10	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	5	0	10
6	8	15	0	0	0	0	0	0	0	1	0	0	0	x	0	0	0	0	0	0	4	-7	
8	6	15	0	0	0	0	0	0	1	0	0	0	0	0	x	0	0	0	0	0	4	-7	
7	9	15	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	0	13	11	
9	7	15	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0	0	0	0	13	11	
10	11	20	0	1	1	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	16	12	
11	10	20	1	0	0	1	0	0	1	0	0	0	1	0	0	0	0	0	0	0	16	12	
12	12	20	0	0	0	0	0	0	0	0	0	0	0	1	x	0	0	0	0	0	4	-12	-4
13	14	24	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	22	20	
14	13	24	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	22	20	
15	16	30	0	0	0	0	0	0	0	1	0	1	0	0	0	0	1	0	0	0	27	24	
16	15	30	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	1	0	0	27	24	
17	18	60	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0	1	51	42	
18	17	60	0	0	0	0	1	0	0	0	0	1	0	0	0	1	1	0	1	0	51	42	
19	19	60	0	0	0	0	1	x	0	0	0	0	0	0	0	1	0	0	1	x	28	-4	52
20	21	64	0	0	1	0	0	0	0	1	0	1	0	x	0	0	1	0	1	0	49	34	
21	20	64	0	0	0	1	0	0	1	0	0	0	1	0	x	0	0	1	0	1	49	34	
22	22	80	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	x	1	x	38	-4	72
23	24	81	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	1	1	x	49	17	
24	23	81	0	0	0	0	0	0	0	0	1	1	0	1	0	0	1	0	1	x	49	17	
25	25	90	0	0	0	0	0	0	0	0	0	0	0	1	x	0	1	x	1	x	37	-16	58

Calculations for E6, E7, E8

“Theorem” (2026)

For E_6, E_7, and E_8, the Lie algebra generated by the reflections

- has a one-dimensional center, plus
- a sum of copies of $\mathfrak{sl}(k)$’s and $\mathfrak{osp}(2m)$’s.
- We can say how many of each type there are, but we can’t say whether an $\mathfrak{osp}(2m)$ is really an $\mathfrak{so}(2m)$ or an $\mathfrak{sp}(2m)$.

Lie superalgebras generated by reflections

arXiv: 2310.01555, 2509.00945

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Consider the group algebra of the symmetric group $\mathbb{C}S_n$. Then there is a corresponding Lie algebra $\mathfrak{L}(S_n)$ defined by

$$[\sigma, \tau] = \sigma \circ \tau - \tau \circ \sigma,$$

where $\sigma, \tau \in S_n$. The structure of $\mathfrak{L}(S_n)$ in terms of simple factors has been considered in this post. One can also ask the same question for the Lie subalgebra of $\mathfrak{L}(S_n)$ generated by transpositions, which was considered in this post.

Now, since there is a $\mathbb{Z}_2$ grading of $\mathbb{C}S_n$, one can also define a Lie superalgebra $s\mathfrak{L}(S_n)$ on it by replacing the commutators with anti-commutators

$$\{\sigma, \tau\} = \sigma \circ \tau + \tau \circ \sigma,$$

for all $\sigma, \tau \in S_n^{(1)}$, where $S_n^{(1)}$ is the odd part of the symmetric group, and all other commutators remain unchanged. Now we have similar questions: what is the structure of $s\mathfrak{L}(S_n)$ in terms of simple Lie superalgebras? What is the subalgebra of $s\mathfrak{L}(S_n)$ generated by transpositions?

My attempt is for $n = 3$, $s\mathfrak{L}(S_n) \cong \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$, while the subalgebra generated by transpositions is $\mathfrak{sl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$. I think in general $s\mathfrak{L}(S_n)$ should be very similar to $\mathfrak{L}(S_n)$, but it might be much harder to determine the subalgebra generated by transpositions.

rt.representation-theory lie-algebras permutations symmetric-groups lie-superalgebras

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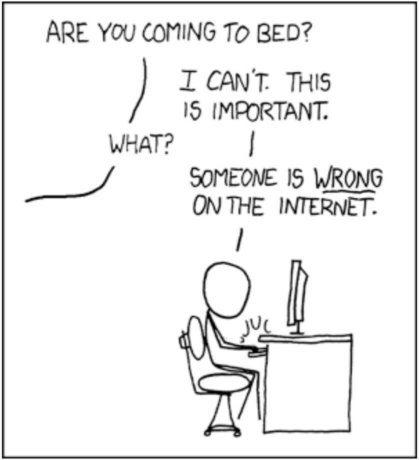
edited Oct 4 at 16:50 Jules Lamers

asked Aug 8, 2022 at 20:26 WunderNatur



https://m.xkcd.com/386/

Supergroups

If $G = (W, \{s_1, \ldots, s_r\})$ is a finite Coxeter system, then G is a supergroup in which the reflections are of odd superdegree.

Example: The symmetric group

- $(\mathfrak{S}_n)_{\bar{0}} = A_n$, the alternating group.
- $(\mathfrak{S}_n)_{\bar{1}} = \mathfrak{S}_n \backslash A_n$, the set of odd permutations.

The $\mathbb{Z}_2$-grading on G extends to superalgebra structure on $\mathbb{C}G$.

Make $\mathbb{C}G$ into a Lie superalgebra via the graded commutator

$$[x, y] = xy - (-1)^{\bar{x} \cdot \bar{y}} yx.$$

Theorem (DK)

Suppose G is a finite (irreducible) Coxeter group. Let $\mathfrak{g}$ be the Lie superalgebra generated by the set S of reflections in G. Then:

$$\mathfrak{g} = \mathfrak{D}(\mathbb{C}G) \oplus \mathbb{C} \cdot T_{short} \oplus \mathbb{C} \cdot T_{long}$$

where $T_{short} = \sum_{s \in S_{short}} s, \quad T_{long} = \sum_{s \in S_{long}} s$, and

$$\mathfrak{D}(\mathbb{C}G) \cong \Big[\bigoplus_{W \in \mathbf{Irr}_Q(G)} \mathfrak{sq}(W) \Big] \oplus \Big[\bigoplus_{W \in \mathbf{Irr}_M(G)} \mathfrak{sl}(W) \Big]$$

$$\mathfrak{sq}(W) \cong \left\{ \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right] \in \mathfrak{gl}(m|m) : \mathbf{tr}(B) = 0 \right\}$$

$$\mathfrak{sl}(W) \cong \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \in \mathfrak{gl}(m|m) : \mathbf{tr}(A) - \mathbf{tr}(D) = 0 \right\}$$