The Lie superalgebra of transpositions

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What is the Lie superalgebra generated by permutations?

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Consider the group algebra of the symmetric group $\mathbb{C}S_n$. Then there is a corresponding Lie algebra $\mathfrak{L}(S_n)$ defined by

$$[\sigma, \tau] = \sigma \circ \tau - \tau \circ \sigma,$$

where $\sigma, \tau \in S_n$. The structure of $\mathfrak{L}(S_n)$ in terms of simple factors has been considered in this post. One can also ask the same question for the Lie subalgebra of $\mathfrak{L}(S_n)$ generated by transpositions, which was considered in this post.

Now, since there is a $\mathbb{Z}_2$ grading of $\mathbb{C}S_n$, one can also define a Lie superalgebra $s\mathfrak{L}(S_n)$ on it by replacing the commutators with anti-commutators

$$\{\sigma, \tau\} = \sigma \circ \tau + \tau \circ \sigma,$$

for all $\sigma, \tau \in S_n^{(1)}$, where $S_n^{(1)}$ is the odd part of the symmetric group, and all other commutators remain unchanged. Now we have similar questions: what is the structure of $s\mathfrak{L}(S_n)$ in terms of simple Lie superalgebras? What is the subalgebra of $s\mathfrak{L}(S_n)$ generated by transpositions?

My attempt is for $n = 3$, $s\mathfrak{L}(S_n) \cong \mathfrak{gl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$, while the subalgebra generated by transpositions is $\mathfrak{sl}(1|1) \oplus \mathfrak{gl}(1|0) \oplus \mathfrak{gl}(0|1)$. I think in general $s\mathfrak{L}(S_n)$ should be very similar to $\mathfrak{L}(S_n)$, but it might be much harder to determine the subalgebra generated by transpositions.

rt.representation-theory lie-algebras permutations symmetric-groups lie-superalgebras

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edited Oct 4 at 16:50 Jules Lamers

asked Aug 8, 2022 at 20:26 WunderNatur

The group algebra of the symmetric group, as a superalgebra

The symmetric group S_n is a supergroup, with

- $(S_n)_{\overline{0}} = A_n$, the alternating group.
- $(S_n)_{\overline{1}} = S_n \backslash A_n$, the set of odd permutations.

This extends to a $\mathbb{Z}_2$-grading on the group algebra $\mathbb{C}S_n$, with

- $(\mathbb{C}S_n)_{\overline{0}} = \mathbb{C}A_n$, the group algebra of the alternating group.

A question of WunderNatur

Question

Considering the group algebra $\mathbb{C}S_n$ of the symmetric group S_n as a superalgebra (by considering the even permutations in S_n to be of even superdegree and the odd permutations in S_n to be of odd superdegree), and considering $\mathbb{C}S_n$ as a Lie superalgebra via the super commutator,

$$[x, y] = xy - (-1)^{\bar{x}\cdot\bar{y}} yx,$$

what is the structure of $\mathbb{C}S_n$ as a Lie superalgebra, and what is the structure of the Lie superalgebra generated by the transpositions?

Super = graded by $\mathbb{Z}/2\mathbb{Z}$ with topologists' sign conventions. Compare super and non-super versions of $[\tau, \tau]$ for τ a transposition.

Classical (non-super) analogue of this question

I. Marin, *L'algèbre de Lie des transpositions*, J. Algebra **310** (2007).

Wedderburn–Artin Theorem

Let A be a finite-dimensional associative semisimple algebra over $\mathbb{C}$, and let $V_1, \ldots, V_m$ be a complete set of pairwise non-isomorphic simple A-modules. Then as a $\mathbb{C}$-algebra,

$$A \cong \mathrm{End}(V_1) \oplus \cdots \oplus \mathrm{End}(V_m).$$

In particular, A is a direct sum of simple $\mathbb{C}$-algebras.

The group algebra of the symmetric group S_n

Given $\lambda \vdash n$, let S^λ be the corresponding Specht module. Then

$$\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} \mathrm{End}(S^\lambda).$$

Thus as a Lie algebra under the commutator, $\mathbb{C}S_n \cong \bigoplus_{\lambda \vdash n} \mathfrak{gl}(S^\lambda)$.

Does this carry over in some way to the superalgebra structure of $\mathbb{C}S_n$?

Semisimple superalgebras

A superalgebra A is semisimple if every A-supermodule V is a direct sum of simple A-supermodules.

Super Wedderburn–Artin Theorem

Let A be a finite-dimensional associative semisimple superalgebra over $\mathbb{C}$. Then A is isomorphic to a product of simple superalgebras.

J. Brundan and A. Kleshchev, Projective representations of symmetric groups via Sergeev duality, Math. Z. **239** (2002), no. 1, 27–68.

S.-J. Cheng and W. Wang, Dualities and representations of Lie superalgebras, Graduate Studies in Mathematics, vol. 144, AMS 2012.

Lemma

Let A be a finite-dimensional associative superalgebra. Then A is semisimple as a superalgebra if and only if A is semisimple as an ordinary ungraded algebra.

So what are the simple superalgebras that occur as factors in $\mathbb{C}S_n$? They come in two flavors...

Type M simple superalgebras

If $V = \mathbb{C}^{m|n}$, then $\mathrm{End}(V) \cong M(m|n)$ is a simple superalgebra, where

$$M(m|n) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] : \begin{array}{ll} A \in M_m(\mathbb{C}), & B \in M_{m\times n}(\mathbb{C}), \\ C \in M_{n\times m}(\mathbb{C}), & D \in M_n(\mathbb{C}). \end{array} \right\}.$$

As an ungraded associative algebra, $M(m|n) \cong \mathfrak{gl}(m+n)$.

Type Q simple superalgebras ("isomeric")

If $V = \mathbb{C}^{n|n}$ with odd involution $J : V \to V$, then

$$Q(V) = \{\theta \in \mathsf{End}(V) : J \circ \theta = \theta \circ J\}$$

is a simple superalgebra. One has $Q(V) \cong Q(n)$, where

$$Q(n) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right] : A \in M_n(\mathbb{C}), B \in M_n(\mathbb{C}) \right\}.$$

As an ungraded associative algebra, $Q(n) \cong \mathfrak{gl}(n) \oplus \mathfrak{gl}(n)$ via the map

$$\left[\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right] \mapsto (A + B, A - B).$$

Two types of simple supermodules

Simple superalgebra summands in $\mathbb{C}S_n$ correspond to isomorphism classes of simple $\mathbb{C}S_n$-supermodules.

Definition

Let V be a simple A-supermodule.

- Say that V is absolutely irreducible (or of Type M) if V is simple as an ungraded A-module.
- Say that V is self-associate (or of Type Q) if V is reducible as an ungraded A-module.

Self-associate simple modules

Let $\pi_V : V \to V$ be the parity automorphism, $\pi_V(v) = (-1)^{\bar{v}} \cdot v$.

Lemma

Let V be a self-associate simple A-supermodule. Then there exists an ungraded simple A-submodule U of V such that

$$V = U \oplus \pi_V(U),$$

with $\pi_V(U)$ also a simple A-submodule, $U \not\cong \pi_V(U)$, and

$$V_{\bar{0}} = \{u + \pi_V(u) : u \in U\}, \quad V_{\bar{1}} = \{u - \pi_V(u) : u \in U\}.$$

An odd involution $J_V : V \to V$ is defined by

$$J_V(u \pm \pi_V(u)) = u \mp \pi_V(u).$$

Structure of semisimple superalgebras

Super Artin–Wedderburn Theorem

Let A be a finite-dimensional semisimple $\mathbb{C}$-superalgebra.

If $\{V_1, \ldots, V_n\}$ is a complete set of simple A-supermodules (up to homogeneous isomorphism), such that $V_1, \ldots, V_m$ are absolutely irreducible and $V_{m+1}, \ldots, V_n$ are self-associate, then

$$A \cong \Big[\bigoplus_{i=1}^{m} \mathrm{End}(V_i) \Big] \oplus \Big[\bigoplus_{i=m+1}^{n} Q(V_i) \Big].$$

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Exercise

Let D_n be the dihedral group of order $2n$, viewed as a supergroup with $(D_n)_{\bar{0}}$ the subgroup of rotations. Work out the superalgebra structure of $\mathbb{C}D_n$.

Simple supermodules for the symmetric group

What do YOU think the simple $\mathbb{C}S_n$-supermodules look like?

Simple supermodules for the symmetric group

What do YOU think the simple $\mathbb{C}S_n$-supermodules look like?

Let $\mathcal{P}(n) = \{\lambda : \lambda \vdash n\}$.

Given $\lambda \vdash n$, let λ' be the conjugate (transpose) partition.

Let $\overline{\mathcal{P}}(n)$ be a fixed set of representatives for the relation $\lambda \sim \lambda'$.

Let $E_n = \left\{\lambda \in \overline{\mathcal{P}}(n) : \lambda \neq \lambda'\right\}$ and $F_n = \left\{\lambda \in \overline{\mathcal{P}}(n) : \lambda = \lambda'\right\}$.

Simple supermodules for $\mathbb{C}S_n$ (up to parity shift)

Simple $\mathbb{C}S_n$-supermodules are indexed by the set $\overline{\mathcal{P}}(n)$.

$$W^\lambda = \begin{cases} S^\lambda \oplus S^{\lambda'} & \text{if } \lambda \in E_n \text{ (Type Q, self-associate case)} \\ S^\lambda & \text{if } \lambda \in F_n \text{ (Type M, absolutely irreducible case)} \end{cases}$$

Structure of simple supermodules for the symmetric group

Type Q simple supermodules $W^\lambda = S^\lambda \oplus S^{\lambda'} \quad (\lambda \neq \lambda')$

The odd involution $J_{W^\lambda} : W^\lambda \to W^\lambda$ can be interpreted as an even isomorphism of $\mathbb{C}S_n$-supermodules

$$W^\lambda \cong \Pi(W^\lambda).$$

Type M simple supermodules $W^\lambda = S^\lambda \quad (\lambda = \lambda')$

As a $\mathbb{C}A_n$-module,

$$S^\lambda = S^{\lambda^+} \oplus S^{\lambda^-},$$

These are the homogeneous subspaces of W^λ. Consequently, W^λ is not even-isomorphic to $\Pi(W^\lambda)$ because $S^{\lambda^+} \not\cong S^{\lambda^-}$ as $\mathbb{C}A_n$-modules.

“Multiplicity free” restriction

Restriction to $\mathbb{C}S_{n-1}$ in terms of Young lattice ordering $\mu \prec \lambda$:

$$W^{\lambda}{\downarrow}_{\mathbb{C}S_{n-1}} \cong \begin{cases} \left[\bigoplus_{\substack{\mu \prec \lambda \\ \mu \neq \mu'}} W^{\mu} \right] \oplus \left[\bigoplus_{\substack{\mu \prec \lambda \\ \mu = \mu'}} W^{\mu} \oplus \Pi(W^{\mu}) \right] & \text{if } \lambda \in E_n, \\ \bigoplus_{\substack{\mu \prec \lambda \\ \mathrm{cont}(\lambda/\mu) \geq 0}} W^{\mu} & \text{if } \lambda \in F_n. \end{cases}$$

Group algebra of the symmetric group, as a superalgebra

Get isomorphisms of associative superalgebras

$$\begin{aligned}\mathbb{C}S_n &\cong \left[\bigoplus_{\lambda\in E_n} Q(W^\lambda)\right] \oplus \left[\bigoplus_{\lambda\in F_n} \mathrm{End}(W^\lambda)\right] \\ &\cong \left[\bigoplus_{\lambda\in E_n} Q(f^\lambda)\right] \oplus \left[\bigoplus_{\lambda\in F_n} M(\tfrac{1}{2}f^\lambda, \tfrac{1}{2}f^\lambda)\right]\end{aligned}$$

where $f^\lambda = \dim(S^\lambda)$. Then as a Lie superalgebra,

$$\mathbb{C}S_n \cong \left[\bigoplus_{\lambda\in E_n} \mathfrak{q}(f^\lambda)\right] \oplus \left[\bigoplus_{\lambda\in F_n} \mathfrak{gl}(\tfrac{1}{2}f^\lambda, \tfrac{1}{2}f^\lambda)\right]$$

Derived Lie superalgebras

Given a Lie superalgebra $\mathfrak{g}$, let $\mathfrak{D}(\mathfrak{g})$ be its derived subsuperalgebra.

$$\begin{aligned}\mathfrak{D}(\mathfrak{gl}(W^\lambda)) &= \mathfrak{sl}(W^\lambda)\\ &\cong \mathfrak{sl}(m|m) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathfrak{gl}(m|m) : \mathrm{tr}(A) - \mathrm{tr}(D) = 0 \right\}\\ \mathfrak{D}(\mathfrak{q}(W^\lambda)) &= \mathfrak{sq}(W^\lambda)\\ &\cong \mathfrak{sq}(n) := \left\{ \left[\begin{array}{c|c} A & B \\ \hline B & A \end{array}\right] \in \mathfrak{q}(n) : \mathrm{tr}(B) = 0 \right\}\end{aligned}$$

Lie superalgebra generated by transpositions

Let $\mathfrak{g}_n \subset \mathbb{C}S_n$ be the Lie superalgebra generated by all transpositions.

Let $T_n = \sum_{1 \le i < j \le n} (i,j) \in \mathbb{C}S_n$ be the sum in $\mathbb{C}S_n$ of all transpositions.

Main Theorem

$$\mathfrak{g}_n = \mathfrak{D}(\mathbb{C}S_n) \oplus \mathbb{C} \cdot T_n,$$

where

$$\mathfrak{D}(\mathbb{C}S_n) \cong \Big[\bigoplus_{\lambda \in E_n} \mathfrak{sq}(W^\lambda) \Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{sl}(W^\lambda) \Big]$$

$\mathfrak{g}_n \subseteq \mathfrak{D}(\mathbb{C}S_n) + \mathbb{C}T_n$ because $\mathfrak{g}_n$ is generated by T_n and the set

$$\left\{ \tau - \frac{2}{n(n-1)} \cdot T_n : \tau \text{ is a transposition} \right\}$$

which is seen to be a subset of $\mathfrak{D}(\mathbb{C}S_n)$. Hard part is showing $\mathfrak{D}(\mathbb{C}S_n) \subseteq \mathfrak{g}_n$.

Ideas behind the proof of the Main Theorem

Let $\mathfrak{g} = \mathfrak{g}_n$. Want $\mathfrak{g} = \Big[\bigoplus_{\lambda \in E_n} \mathfrak{sq}(W^\lambda)\Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{sl}(W^\lambda)\Big] \oplus \mathbb{C} \cdot T_n$.

1. Show by induction on n (and brutish force) that

$$\operatorname{im}\big(\mathfrak{g} \to \operatorname{End}(W^\lambda)\big) = \begin{cases} \mathfrak{sq}(W^\lambda) + \mathbb{C} \cdot (\operatorname{cont}(\lambda) \cdot J_{W^\lambda}) & \text{if } \lambda \in E_n, \\ \mathfrak{sl}(W^\lambda) & \text{if } \lambda \in F_n. \end{cases}$$

Use description of the restrictions $W^\lambda\downarrow_{\mathbb{C}S_{n-1}}$, and Gelfand–Zeitlin bases for the S^λ given by the simultaneous eigenvectors for the action of the Jucys–Murphy elements.

2. Deduce $\mathfrak{g}_{\overline{0}}$ is reductive, hence $\mathfrak{D}(\mathfrak{g}_{\overline{0}})$ is a semisimple Lie algebra.

Observe by 1 that each W^λ is a semisimple $\mathfrak{g}_{\overline{0}}$-module. Then $\bigoplus_{\lambda \in E_n \cup F_n} W^\lambda$ is a faithful, finite-dimensional, completely reducible $\mathfrak{g}_{\overline{0}}$-module, so $\mathfrak{g}_{\overline{0}}$ is reductive.

Ideas behind the proof of the Main Theorem

3. Show that $\mathfrak{D}(\mathfrak{g}_{\bar{0}})$ is as big as it should be:

$$\mathfrak{D}(\mathfrak{g}_{\bar{0}}) = \Big[\bigoplus_{\lambda \in E_n} \mathfrak{sl}(W_\lambda)\Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{sl}(W^\lambda_{\bar{0}}) \oplus \mathfrak{sl}(W^\lambda_{\bar{1}})\Big]$$

By semisimplicity, $\mathfrak{D}(\mathfrak{g}_{\bar{0}})$ is a direct sum of special linear Lie algebras; need to show all factors are distinct. Argument uses the facts

– for $n \geq 5$, $\mathfrak{D}(\mathfrak{g}_{\bar{0}})$ can distinguish simple $\mathbb{C}A_n$-modules. Note that $(i\,j)(k\,\ell) \in \mathfrak{D}(\mathfrak{g}_{\bar{0}})$ if i, j, k, ℓ are distinct, and elements of this form generate $\mathbb{C}A_n$ as an associative algebra.

– W^λ and W^μ have simple $\mathbb{C}A_n$-submodules in common only if $\lambda = \mu$.

4. Apply semisimple action of $\mathfrak{D}(\mathfrak{g}_{\bar{0}})$ to deduce that $\mathfrak{D}(\mathbb{C}S_n)_{\bar{1}} \subseteq \mathfrak{g}_{\bar{1}}$. Since $\mathfrak{D}(\mathbb{C}S_n)_{\bar{1}}$ generates essentially all of $\mathfrak{D}(\mathbb{C}S_n)$, then $\mathfrak{D}(\mathbb{C}S_n) \subseteq \mathfrak{g}$.

End result (once more)

Main Theorem

$$\mathfrak{g}_n = \mathfrak{D}(\mathbb{C}S_n) \oplus \mathbb{C} \cdot T_n,$$

where

$$\mathfrak{D}(\mathbb{C}S_n) \cong \Big[\bigoplus_{\lambda \in E_n} \mathfrak{sq}(W^\lambda)\Big] \oplus \Big[\bigoplus_{\lambda \in F_n} \mathfrak{sl}(W^\lambda)\Big]$$

Then $\dim(\mathfrak{g}_n) = n! - |E_n \cup F_n| + 1.$

L'algèbre de Lie des transpositions, J. Algebra 310 (2007)

Marin studied the classical (non-super) analogue of this problem, motivated by the representation theory of the braid group.

Proposition 1. *L'algèbre de Lie* $\mathfrak{g}_n$ *est réductive, et son centre est de dimension* 1*, engendré par la somme* T_n *de toutes les transpositions. En conséquence* $\mathfrak{g}_n \simeq \Bbbk \times \mathfrak{g}'_n$*, et l'image de* $\mathfrak{g}_n$ *dans* $\mathfrak{gl}(\lambda)$ *est* $\mathfrak{g}_\lambda \subset \mathfrak{sl}(\lambda)$ *si* T_n *agit par* 0*, et* $\Bbbk \times \mathfrak{g}_\lambda$ *sinon.*

Marin can deduce right off the bat that $\mathfrak{g}_n$ is reductive.

Théorème A. *Pour tout* $n \geqslant 3$, ϕ_n *est surjectif. En particulier,*

$$\mathfrak{g}'_n \simeq \mathfrak{sl}_{n-1}(\Bbbk) \times \left(\prod_{\lambda \in E_n/\sim} \mathfrak{sl}(\lambda) \right) \times \left(\prod_{\lambda \in F_n} \mathfrak{osp}(\lambda) \right)$$

et les représentations ρ_λ *de* $\mathfrak{g}'_n$ *sont deux à deux non isomorphes.*

Marin's $E_n/\sim$ and F_n don't include any hook partitions. His $\mathfrak{osp}$ is the French notation for "$\mathfrak{o}$ or $\mathfrak{sp}$."

Overall, the Lie algebra of transpositions is less than half the size of the Lie superalgebra of transpositions.

May 19, 2010

May 21, 2010