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On Strongly Interacting Internal Solitary Waves

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ABSTRACT. *The Cauchy problem and global well-posedness for a mathematical model of the strong interaction of two-dimensional, long, internal gravity waves propogating on neighboring pycnoclines in a stratified fluid have been studied by Bona, Ponce, Saut, Tom, and others. We show that global well-posedness occurs even when the initial data is rough.*

1. Introduction

This paper is concerned with the initial-value problem

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + uu_x + a_1 vv_x + a_2(uv)_x = 0 \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + vv_x + b_2 a_2 uu_x + b_2 a_1 (uv)_x + rv_x = 0 \\ u(x,0) = u_0(x) \\ v(x,0) = v_0(x), \end{cases} \tag{1}$$

where a_1, a_2, a_3, b_1, b_2, and r are real constants with b_1, b_2 positive; $u = u(x,t), v = v(x,t)$ are real-valued functions of the two real variables x and t; and subscripts adorning u and v denote partial differentiation. This system has the structure of a pair of Korteweg–de Vries equations coupled through both dispersive and nonlinear effects. It was derived by Gear and Grimshaw [GG] as a model to describe the strong interaction of weakly nonlinear, long waves. System (1) was studied extensively by Bona, Ponce, Saut, and Tom [BPST] and we strongly urge the reader to refer to that paper for an excellent and extensive discussion of the physical significance of (1) and for further references. As in [BPST], we will assume that $r = 0$. However, we will only assume that $|a_3| \neq 1/\sqrt{b_2}$ rather than $|a_3| < 1/\sqrt{b_2}$. Problem (1) is said to be *globally well-posed* in $L^2(R) \times L^2(R)$ if existence, uniqueness, persistence, and continuous dependence on the initial data occur for all $t \geq 0$. We will make use of only one of the four known conservation laws satisfied by solutions of (1). This law asserts that when (u, v) is a solution to (1), the quantity $\int_{-\infty}^{\infty} (b_2 u^2 + b_1 v^2)\, dx$ is an invariant sometimes called the energy.

The natural definitions $|(u,v)| = (b_2 u^2 + b_1 v^2)^{\frac{1}{2}}$, $\|(u,v)\|_p = (\int_{-\infty}^{\infty} |(u,v)|^p\, dx)^{\frac{1}{p}}$ for $1 \leq p < \infty$ and $\|(u,v)\|_\infty = \text{Ess sup}\, |(u,v)|$ allow us to rewrite the energy conservation law as

$$\|(u,v)\|_2^2 = \text{const.} \tag{2}$$

For $b \in R$, Kenig, Ponce, and Vega [KPV1] defined the Sobolev spaces X_b as the completion of the Schwartz space $\mathcal{S}(R^2)$ with respect to the norm

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$$\|f\|_{X_b} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\hat{f}(\xi, \tau)|^2 \left(1 + |\tau - \xi^3|\right)^{2b} d\xi\, d\tau \right)^{\frac{1}{2}},$$

where $\hat{u}(\xi, \tau) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(x, t) e^{-i(x\xi + t\tau)}\, dx\, dt$. Our two-dimensional version of this is X_b^2, the completion of the Schwartz space $\mathcal{S}(R^2)$ with respect to the norm

$$\|(u, v)\|_{X_b^2} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |(\hat{u}(\xi, \tau), \hat{v}(\xi, \tau))|^2 \left(1 + |\tau - \xi^3|\right)^{2b} d\xi\, d\tau \right)^{\frac{1}{2}}.$$

In the following, we use the symbol c to denote a generic constant so that the value of c can change, even when passing from one side of an inequality to the other.

Much recent work was done on the Korteweg–de Vries equation by Bourgain [B] and Kenig, Ponce, and Vega [KPV1, KPV2]. Our methods will be those of Kenig, Ponce, and Vega [KPV1, KPV2]. We are grateful to Carlos Kenig for having suggested this problem and the means for its solution.

We adapt the notation of [BPST]. If $[0, T]$ is an interval and X is a Banach space with norm $\|\cdot\|_X$, then

$$L^p([0, T] : X) = \left\{ u : [0, T] \to X \text{ such that } \|u\|^p_{L^p([0,T]:X)} = \int_0^T \|u\|_X^p < \infty \right\}.$$

The space $C([0, T] : X)$ consists of all continuous functions mapping from $[0, T]$ into X. Since $[0, T]$ is a compact set, $C([0, T] : X)$ is a Banach space when equipped with the norm $\|\cdot\|_{L^\infty([0,T]:X)}$.

2. Results

As mentioned in the introduction, we are studying the problem

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} + u u_x + a_1 v v_x + a_2 (uv)_x = 0 \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx} + v v_x + b_2 a_2 u u_x + b_2 a_1 (uv)_x = 0 \\ u(x, 0) = u_0(x) \\ v(x, 0) = v_0(x), \end{cases} \tag{3}$$

where a_1, a_2, a_3, b_1, and b_2 are real constants with b_1 and b_2 positive and $|a_3| \neq 1/\sqrt{b_2}$. Our main result is as follows.

Theorem 1.

For any $(u_0, v_0) \in L_x^2(R) \times L_x^2(R)$, there is a unique solution of system (3) *with (u_0, v_0) as initial data such that*

$$(u, v) \in C([0, \infty) : L_x^2(R) \times L_x^2(R)), \tag{4}$$

and

$$u \text{ and } v \text{ are in } L^p_{x,\text{loc}}(R : L^2_{t,\text{loc}}(R)), \text{ for } 1 \leq p \leq \infty. \tag{5}$$

We begin the proof of this by starting with a more modest and local version.

Theorem 2.

There is an $\epsilon_0 > 0$ *such that for any* $(u_0, v_0) \in L_x^2(R) \times L_x^2(R)$ *with* $\|(u_0, v_0)\|_{L_x^2 \times L_x^2} < \epsilon_0$, *there is a unique solution of system* (3) *on* $[0, \frac{1}{2}]$ *with* (u_0, v_0) *as initial data such that*

$$(u, v) \in C\left(\left[0, \frac{1}{2}\right] : L_x^2(R) \times L_x^2(R)\right), \tag{6}$$

and

$$(u, v) \in X_b^2 \text{ for any } b \in \left(\frac{1}{2}, \frac{3}{4}\right). \tag{7}$$

So that, in particular,

$$u \text{ and } v \text{ are in } L_{x,\mathrm{loc}}^p(R : L_t^2(R)), \text{ for } 1 \le p \le \infty. \tag{8}$$

Furthermore,

$$uu_x, uv_x, u_xv, \text{ and } vv_x \in X_{b-1}, \tag{9}$$

and (u_t, v_t) *satisfies*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |(\hat{u}_t, \hat{v}_t)|^2 (1 + |\xi|)^{-6} \left(1 + |\tau - \xi^3|\right)^{2b-2} d\xi\, d\tau < \infty. \tag{10}$$

There exists a neighborhood N *of* $(u_0, v_0) \in L_x^2(R) \times L_x^2(R)$ *such that for* $(u_0^*, v_0^*) \in N$, *there exists a solution* (u^*, v^*) *solving system* (3) *and satisfying the properties* (6)–(10) *on* $[0, \frac{1}{2}]$ *and the map* $(u_0^*, v_0^*) \to (u^*, v^*)$ *is Lipschitz (in the sense expressed by inequality* (26)).

Remark. Symmetry considerations allow us to restrict our proofs to the case of future time only. More explicitly, suppose system (3) can be solved for all $t \ge 0$ for any $L_x^2 \times L_x^2$ initial data. If $(u(x, t), v(x, t))$ is a solution when $t \ge 0$ corresponding to initial data $(p_0(-x), q_0(-x))$, then $(u(-x, -t), v(-x, -t))$ is a solution to system (3) with initial data $(p_0(x), q_0(x))$. □

To prove Theorem 2, when $a_3 \neq 0$ we will need to diagonalize

$$\begin{cases} u_t + u_{xxx} + a_3 v_{xxx} \\ b_1 v_t + v_{xxx} + b_2 a_3 u_{xxx}, \end{cases} \tag{11}$$

which is the linear part of our system by making a change of variables essentially introduced in [BPST]. This will change system (3) into a pair of equations with decoupled linear parts. Let

$$\begin{pmatrix} p(x,t) \\ q(x,t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(1 - \frac{1-b_1}{\lambda b_1}\right) u(\alpha_+^{1/3}x, t) + \frac{a_3}{\lambda} v(\alpha_+^{1/3}x, t) \\ \frac{1}{2}\left(1 + \frac{1-b_1}{\lambda b_1}\right) u(\alpha_-^{1/3}x, t) - \frac{a_3}{\lambda} v(\alpha_-^{1/3}x, t) \end{pmatrix},$$

where $\lambda = \sqrt{(1 - 1/b_1)^2 + 4b_2 a_3^2/b_1}$ and $\alpha_\pm = \frac{1}{2}\{1 + 1/b_1 \pm \lambda\}$. Note that the condition $|a_3| \neq 1/\sqrt{b_2}$ guarantees that α_+ and α_- are not 0. Under this change of variables, the terms (11) become

$$\begin{cases} p_t + p_{xxx} \\ q_t + q_{xxx} \end{cases}$$

and (3) become

$$
\begin{cases}
(p_t + p_{xxx})(x,t) + m(x,t) = 0 \\
(q_t + q_{xxx})(x,t) + n(x,t) = 0 \\
p(x,0) = \frac{1}{2}\left(1 - \frac{1-b_1}{\lambda b_1}\right) u_0(\alpha_+^{1/3} x) + \frac{a_3}{\lambda} v_0(\alpha_+^{1/3} x) \\
q(x,0) = \frac{1}{2}\left(1 + \frac{1-b_1}{\lambda b_1}\right) u_0(\alpha_-^{1/3} x) - \frac{a_3}{\lambda} v_0(\alpha_-^{1/3} x),
\end{cases}
\tag{12}
$$

where $m(p,q,x,t) = m(x,t)$ is defined to be

$$
\left(\frac{1-\alpha_-}{\lambda}(uu_x + a_1 vv_x + a_2(uv)_x) + \frac{a_3}{b_1\lambda}(vv_x + b_2a_2uu_x + b_2a_1(uv)_x)\right)(\alpha_+^{1/3} x, t), \tag{13}
$$

and $n(p,q,x,t) = n(x,t)$ is defined to be

$$
\left(\frac{\alpha_+ - 1}{\lambda}(uu_x + a_1 vv_x + a_2(uv)_x) - \frac{a_3}{b_1\lambda}(vv_x + b_2a_2uu_x + b_2a_1(uv)_x)\right)(\alpha_-^{1/3} x, t), \tag{14}
$$

and we think of p and q as the unknown functions and m and n as abbreviations of combinations of p and q via the inverse formulas

$$
\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \begin{pmatrix} p\left(\frac{x}{\alpha_+^{1/3}}, t\right) + q\left(\frac{x}{\alpha_-^{1/3}}, t\right) \\ \frac{1}{2a_3}\left(\frac{1-b_1}{b_1} + \lambda\right) p\left(\frac{x}{\alpha_+^{1/3}}, t\right) + \frac{1}{2a_3}\left(\frac{1-b_1}{b_1} - \lambda\right) q\left(\frac{x}{\alpha_-^{1/3}}, t\right) \end{pmatrix}. \tag{15}
$$

Some notation will be useful. Let $P = (p,q)$, $U = (u,v)$, and $M = (m,n)$. Also let $U_0(x) = (u_0(x), v_0(x))$ and

$$
P_0(x) = \frac{1}{2}\left(\left(1 - \frac{1-b_1}{\lambda b_1}\right) u_0(\alpha_+^{1/3} x) + \frac{a_3}{\lambda} v_0(\alpha_+^{1/3} x), \left(1 + \frac{1-b_1}{\lambda b_1}\right) u_0(\alpha_-^{1/3} x) - \frac{a_3}{\lambda} v_0(\alpha_-^{1/3} x)\right).
$$

Then system (12) can be written

$$
\begin{aligned}
P_t + P_{xxx} &= -M \\
P(x,0) &= P_0(x).
\end{aligned}
\tag{16}
$$

If $M = 0$, system (16) becomes a linear homogeneous system. As is well known and easy to verify, in that case the solution is given by $W(t)P_0 = \int_{-\infty}^{\infty} e^{i(t\xi^3 + x\xi)} \tilde{P}_0(\xi)\, d\xi$, where the integral of an R^2-valued function is given by $\int_{-\infty}^{\infty}(p,q) = (\int_{-\infty}^{\infty} p, \int_{-\infty}^{\infty} q)$, $\tilde{P} = (\tilde{p}, \tilde{q})$, and $\tilde{p}(\xi)$ denotes $\frac{1}{2\pi}\int_{-\infty}^{\infty} p(x)e^{-ix\xi}\, dx$, the Fourier transform of p with respect to the space variable x. (As in the introduction, we reserve the circumflex symbol for a full two-dimensional Fourier transform in both space and time.)

It is easy to see that to solve the nonlinear problem (16) in X_b^2, we need only find a function P so that the integral equation

$$
P(t) = W(t)P_0 - \int_0^t W(t-\tau)M(\tau)\, d\tau \tag{17}
$$

holds in the space X_b^2. Define an operator Φ by $\Phi(P) = W(t)P_0 - \int_0^t W(t-\tau)M(\tau)\, d\tau$. Then (17) holds exactly when P is a fixed point of the operator Φ. A standard way to produce a fixed point is to iterate a contraction. So, our goal is now to show that Φ, when restricted to a small ball in X_b^2, is a contraction. We will achieve this goal in two steps. First, we will show that for some sufficiently small $r > 0$, Φ maps $X_b^2(r)$, the origin-centered ball of radius r in X_b^2, *into* itself. Second, we will show that Φ restricted to the ball $X_b^2(r)$ is a contraction, i.e., that there is a $\rho < 1$ such that whenever P and P' are in $X_b^2(r)$, $\|\Phi(P) - \Phi(P')\|_{X_b^2} \le \rho \|P - P'\|_{X_b^2}$.

To carry out this program we need several lemmas. Let $\theta(t)$ be a C^∞ function from R to $[0, 1]$, supported in $(-2, 2)$ and identically 1 on the interval $[-1, 1]$. Let $\psi(t)$ be a C^∞ function from R to $[0, 1]$, supported in $(-2, 2)$ and identically 1 on the support of θ. The index b will always satisfy $b > \frac{1}{2}$ and the parameter δ will always belong to $(0, 1]$. The function space $L_x^2(R) \times L_x^2(R)$ will be denoted as L_X^2.

Lemma 3.
$$\|\theta(t/\delta)W(t)U_0\|_{X_b^2} \le c\delta^{(1-2b)/2}\, \|U_0\|_{L_X^2}.$$

Lemma 4.
$$\|\theta(t/\delta)U\|_{X_b^2} \le c\delta^{(1-2b)/2}\, \|U\|_{X_b^2}.$$

Lemma 5.
$$\|\theta(t/\delta) \textstyle\int_0^t W(t-t')U(x,t')\,dt'\|_{X_b^2} \le c\delta^{(1-2b)/2}\, \|U\|_{X_{b-1}^2}.$$

These three lemmas are essentially Lemmas 3.1 to 3.3 (with $s = 0$) from pages 7 to 9, respectively, of [KPV1]. Their proofs are very similar to the proofs given in that paper. Actually, our use of Lemmas 3 and 5 in this paper will be restricted to the special case of $\delta = 1$. The following two lemmas are similar to Corollary 2.7 and Lemma 3.3 of [KPV2].

Lemma 6.
Let $\frac{1}{2} < b \le b' \le \frac{3}{4}$. *Then*
$$||M||_{X_{b'-1}^2} \le c\left(||P||_{X_b^2}\right)^2.$$

Lemma 7.
Let $\frac{1}{2} < b \le b' \le \frac{3}{4}$. *Then for* $\delta \in (0, 1)$,
$$||\psi(t/\delta)U||_{X_{b-1}^2} \le c\delta^{\eta}||U||_{X_{b'-1}^2},$$
where $\eta = \frac{(b'-b)}{4b'}$.

In fact, Corollary 2.7 of [KPV2] proclaims that for $\frac{1}{2} < b \le b' \le \frac{3}{4}$,
$$||uu_x||_{X_{b'-1}} \le c\left(||u||_{X_b}\right)^2.$$
But the same proof shows that
$$||\,(uv)_x\,||_{X_{b'-1}} \le c||u||_{X_b}||v||_{X_b}. \tag{18}$$
To see this, observe that analogous to (2.51) of [KPV2] we have
$$\left\| \frac{\xi}{(1+|\tau-\xi^3|)^{1-b'}} \int\!\!\int \frac{f(\xi_1,\tau_1)}{(1+|\tau_1-\xi_1^3|)^b} \frac{g(\xi-\xi_1,\tau-\tau_1)}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^b}\, d\xi_1\, d\tau_1 \right\|_{L_\xi^2 L_\tau^2}$$
$$\le \left\| \frac{\xi}{(1+|\tau-\xi^3|)^{1-b'}} \left(\int\!\!\int \frac{1}{(1+|\tau_1-\xi_1^3|)^{2b}} \frac{1}{(1+|\tau-\tau_1-(\xi-\xi_1)^3|)^{2b}}\, d\xi_1\, d\tau_1\right)^{\frac{1}{2}} \right\|_{L_\xi^\infty L_\tau^\infty}$$
$$\times \left\| \int\!\!\int |f(\xi_1,\tau_1)|^2\, |g(\xi-\xi_1,\tau-\tau_1)|^2\, d\xi_1\, d\tau_1 \right\|_{L_\xi^2 L_\tau^2},$$

where $f(\xi,\tau)=(1+|\tau-\xi^3|)^b\hat{u}(\xi,\tau)$ and $g(\xi,\tau)=(1+|\tau-\xi^3|)^b\hat{v}(\xi,\tau)$. Lemma 2.4 of [KPV2] asserts that the first norm on the right-hand side is bounded by a constant depending only on b and b'. Consequently, the entire right-hand side is less than or equal to $c||f||_{L^2_\xi L^2_\tau}||g||_{L^2_\xi L^2_\tau}$. This establishes inequality (18).

From (13) and (14) we see that all the terms in M and N are of the form $(uu)_x$, $(vv)_x$; or $(uv)_x$; so Lemma 6 follows from inequality (18).

From the above two lemmas, we get the following result.

Lemma 8.
$\|\psi^2\left(\frac{t}{\delta}\right)M\|_{X^2_{b-1}}\le c\delta^{\theta_0}\left(\|P\|_{X^2_b}\right)^2$ *for some* $\theta_0>0$ *if* $b\in\left(\frac{1}{2},\frac{3}{4}\right)$ *and* $0<\delta<1$.

Proof. Choose $\frac{1}{2}<b<b'\le\frac{3}{4}$. Using Lemma 7, we have

$$\left\|\psi^2\left(\frac{t}{\delta}\right)M\right\|_{X^2_{b-1}}\le c\delta^{\theta_0}||M||_{X^2_{b'-1}}$$

where $\theta_0=\frac{(b'-b)}{4b'}$. By Lemma 6,

$$\|M\|_{X^2_{b'-1}}\le c\left(\|P\|_{X^2_b}\right)^2.$$

The lemma follows by combining the above two inequalities. □

Our last lemma essentially appears as the fourth inequality on page 20 of [KPV1].

Lemma 9.
$\|\int_{t'}^{t}W(t-\tau)M(x,\tau)d\tau\|_{L^2_X}\le c\|M\|_{X^2_{b-1}}$ *for* $0\le t'<t\le 1$.

Proof. It follows from the definition of $W(t)$ that

$$\int_{t'}^{t}W(t-\tau)M(x,\tau)\,d\tau=-i\int\int e^{ix\xi}\hat{M}(\xi,\lambda)e^{i\lambda t'}\frac{e^{i(t-t')\lambda}-e^{i(t-t')\xi^3}}{\lambda-\xi^3}\,d\xi\,d\lambda.$$

We have

$$\begin{aligned}\left\|\int_{t'}^{t}W(t-\tau)M(x,\tau)\,d\tau\right\|_{L^2_X}&=\left\|\int\int e^{ix\xi}\hat{M}(\xi,\lambda)e^{i\lambda t'}\frac{e^{i(t-t')\lambda}-e^{i(t-t')\xi^3}}{\lambda-\xi^3}\,d\xi\,d\lambda\right\|_{L^2_X}\\&=\left\|\int\hat{M}(\xi,\lambda)e^{i\lambda t'}\frac{e^{i(t-t')\lambda}-e^{i(t-t')\xi^3}}{\lambda-\xi^3}\,d\lambda\right\|_{L^2_\Xi}\\&\le 3\left\|\int\left|\hat{M}(\xi,\lambda)\right|\frac{1}{1+|\lambda-\xi^3|}\,d\lambda\right\|_{L^2_\Xi},\end{aligned}$$

where the last line follows from the inequality

$$\left|\frac{e^{i(t-t')\lambda}-e^{i(t-t')\xi^3}}{\lambda-\xi^3}\right|\le\frac{3}{1+|\lambda-\xi^3|}.$$

Denoting $1+|\lambda-\xi^3|$ by $\Delta(\xi,\lambda)$ and using the Cauchy-Schwarz inequality, we have

$$\left\| \int_{t'}^{t} W(t-\tau)M(x,\tau)\,d\tau \right\|_{L^2_X} \le 3 \left\| \int \left|\hat{M}(\xi,\lambda)\right| \Delta^{b-1}(\xi,\lambda) \frac{1}{\Delta^b(\xi,\lambda)}\,d\lambda \right\|_{L^2_\Xi}$$

$$\le 3\sqrt{\int \left(\int \left|\hat{M}(\xi,\lambda)\right|^2 \Delta^{2b-2}(\xi,\lambda)\,d\lambda\right)\left(\int \frac{1}{\Delta^{2b}(\xi,\lambda)}\,d\lambda\right) d\xi}$$

$$\le c\,\|M\|_{X^2_{b-1}}. \qquad \square$$

Proof of Theorem 2. For the duration of this proof, we assume that $t \in [0, \frac{1}{2}]$. First we show that Φ maps $X^2_b(r)$ into itself. Since $\theta(t) = 1$ for $t \in [0, \frac{1}{2}]$, we may write

$$\Phi(P) = \theta(t)W(t)(P_0) - \theta(t)\int_0^t W(t-\tau)\psi^2(\tau)M(x,\tau)\,d\tau.$$

By Lemmas 3 and 5, with $\delta = 1$, we have that

$$\|\Phi(P)\|_{X^2_b} \le c\,\|P_0\|_{L^2_X} + c\|\psi^2 M\|_{X^2_{b-1}};$$

whence by Lemma 6, with $b' = b$,

$$\|\Phi(P)\|_{X^2_b} \le c\,\|P_0\|_{L^2_X} + c\left(\|\psi P\|_{X^2_b}\right)^2.$$

By Lemma 4, with θ replaced by ψ and $\delta = 1$, this becomes

$$\|\Phi(P)\|_{X^2_b} \le c\,\|P_0\|_{L^2_X} + c\left(\|P\|_{X^2_b}\right)^2.$$

Now pick $r > 0$ so small that

$$cr^2 < \frac{1}{2}r. \tag{19}$$

Then pick $\epsilon_0 > 0$ so small that $\|P_0\|_{L^2_X} < \epsilon_0$ implies that

$$c\,\|P_0\|_{L^2_X} < \frac{1}{2}r. \tag{20}$$

Then $\|\Phi(P)\|_{X^2_b} \le c\,\|P_0\|_{L^2_x} + cr^2 < r$ when $\|P\|_{X^2_b} < r$, so that indeed Φ maps $X^2_b(r)$ into itself.

Next, let P and P' be points of $X^2_b(r)$. To show that Φ is a contraction we need to show that $\|\Phi(P) - \Phi(P')\|_{X^2_b} \le \rho\|P - P'\|_{X^2_b}$ for some $\rho < 1$. Now

$$\left\|\Phi(P) - \Phi(P')\right\|_{X^2_b} = \left\|\theta(t)\int_0^t W(t-\tau)\left\{\psi^2(\tau)\left(M(x,\tau) - M'(x,\tau)\right)\right\}d\tau\right\|_{X^2_b},$$

so that, by Lemma 5,

$$\left\|\Phi(P) - \Phi(P')\right\|_{X^2_b} \le c\left\|\psi^2\left(M - M'\right)\right\|_{X^2_{b-1}}. \tag{21}$$

But $M - M' = (m - m', n - n')$. Consider $m - m'$. This is a linear combination of $(uv)_x - (u'v')_x$ and two similar terms. Writing this as $[(u-u')v]_x + [u'(v-v')]_x$, applying first inequality (18) with $b' = b$ and then Lemma 4 with $\delta = 1$, we see that the right-hand side of inequality (21) is controlled by a linear combination of terms such as $c\|u - u'\|_{X_b}\|v\|_{X_b}$. From this it follows that

$$\left\|\psi^2(t')\left(M - M'\right)\right\|_{X^2_{b-1}} \le c\left\|P - P'\right\|_{X^2_b}\left(\|P\|_{X^2_b} + \left\|P'\right\|_{X^2_b}\right) \le 2rc\left\|P - P'\right\|_{X^2_b}.$$

Choose r again so that besides condition (19), $\rho = 2rc < 1$ also holds. (Notice that the choice of ϵ_0 must be similarly modified to keep inequality (20) valid.) We get $\|\Phi(P) - \Phi(P')\|_{X^2_b} \le \rho\|P - P'\|_{X^2_b}$, as desired. In general, Lemma 3 and the above argument also show that if $\|P_0\|_{L^2_X} \le \epsilon_0$, $\|P_0'\|_{L^2_X} \le \epsilon_0$ and $\|P\|_{X^2_b} \le r$, $\|P'\|_{X^2_b} \le r$ and we emphasize the dependence of Φ on P_0 by writing $\Phi(P) = \Phi_{P_0}(P)$, then

$$\left\|\Phi_{P_0}(P) - \Phi_{P_0'}(P')\right\|_{X^2_b} \le c\|P_0 - P_0'\|_{L^2_X} + \rho\|P - P'\|_{X^2_b}. \tag{22}$$

Since Φ is a contraction, there is a unique fixed point, i.e., a pair $P = (p, q) \in X^2_b(r)$ such that, when $t \in [0, \frac{1}{2}]$, $P = \Phi(P) = W(t)(P_0) - \int_0^t W(t - \tau)M(x, \tau)\, d\tau$. From this we see that P satisfies (16). This shows P is a solution. Because Φ is a contraction on $X^2_b(r)$, P is the unique solution to (16) in $X^2_b(r)$, but it is still conceivable that there is another solution $P^* = (p^*, q^*)$ with X^2_b norm exceeding r. However, this is impossible. We will prove this later when we prove Theorem 1.

We now show the solution we derived from the above contraction mapping satisfies the properties listed in the theorem. We begin with (6). Let $0 < t' < t$ and $t - t' < \delta$. Let θ, ψ be the C^∞ functions defined above. We have

$$\|P(x, t) - P(x, t')\|_{L^2_X} \le \|W(t)(P(x, t) - P(x, t'))\|_{L^2_X} + \left\|\int_{t'}^{t} W(t - \tau)\psi^2\left(\frac{\tau - t'}{\delta}\right) M(x, \tau)\, d\tau\right\|_{L^2_X}$$

But W is a continuous operator on L^2_X, so, as δ tends to 0, the first term on the right is $o(1)$. Lemma 9 gives

$$\|P(x, t) - P(x, t')\|_{L^2_X} \le o(1) + c\left\|\psi^2\left(\frac{(\cdot - t')}{\delta}\right) M\right\|_{X^2_{b-1}}.$$

Applying Lemma 6, we have

$$\|P(x, t) - P(x, t')\|_{L^2_X} \le o(1) + c\delta^{\theta_0}\|P\|^2_{X^2_b}.$$

Since $\theta_0 > 0$, this tends to 0 as δ tends to zero. Thus, assertion (6) is proved.

Assertion (7) is obvious.

Since the spaces $L^p_{x,\mathrm{loc}}(R : L^2_t(R))$ are nested and become smaller with increasing p, assertion (8) follows from the containment

$$L^\infty_x(R : L^2_t(R)) \supset X_b. \tag{23}$$

To see this, use Plancherel's theorem and the Cauchy–Schwarz inequality to get

$$
\begin{aligned}
\|u\|^2_{L^\infty_x L^2_t} &= \operatorname*{Essup}_x \int \left| \int e^{-ix\xi} \hat{u}(\xi,\tau)\, d\xi \right|^2 d\tau \\
&\le \int \left(\int |\hat{u}(\xi,\tau)|\, d\xi \right)^2 d\tau \\
&= \int \left(\int \left(1+|\tau-\xi^3|\right)^{-b} |\hat{u}(\xi,\tau)| \left(1+|\tau-\xi^3|\right)^{b} d\xi \right)^2 d\tau \\
&\le \operatorname*{Essup}_\tau \int \left(1+|\tau-\xi^3|\right)^{-2b} d\xi\, \|u\|^2_{X^2_b}.
\end{aligned}
\tag{24}
$$

Taking square roots proves the containment (23). Note that since $b > \frac{1}{2}$,

$$
\begin{aligned}
3\int \left(1+|\tau-\xi^3|\right)^{-2b} d\xi &= \int (1+|y|)^{-2b}(\tau-y)^{-\frac{2}{3}}\, dy \\
&\le \int\limits_{|\tau-y|\le 1} (\tau-y)^{-\frac{2}{3}}\, dy + \int\limits_{|\tau-y|\ge 1} (1+|y|)^{-2b}\, dy,
\end{aligned}
$$

which shows that the supremum on the last line of the estimates (24) is finite.

Lemma 6 implies (9). Using (3) and (9), (10) follows if we can show

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\xi|^6 |\hat{U}|^2 (1+|\xi|)^{-6}(1+|\tau-\xi^3|)^{2b-2}\, d\xi\, d\tau < \infty.
$$

But this is an easy consequence of $U \in X^2_b$. Finally, let P'_0 be an initial data such that $\|P'_0\|_{L^2_x} < \epsilon_0$. Let N be a neighborhood of P'_0 such that $Q'_0 \in N$ implies $\|Q'_0\|_{L^2_x} < \epsilon_0$. Let P' and Q' be the solutions of (16) derived from the contraction mapping with initial conditions P'_0 and Q'_0, respectively. Since $P' = \Phi_{P'_0}(P')$ and $Q' = \Phi_{Q'_0}(Q')$, it follows from inequality (22) that

$$
\|P'-Q'\|_{X^2_b} \le \frac{c}{(1-\rho)} \|P'_0 - Q'_0\|_{L^2_x}. \tag{25}
$$

Furthermore, since $W(t)$ is an isometry, by Lemma 9 we have for any $t \in [0, \frac{1}{2}]$

$$
\begin{aligned}
\|P'(x,t)-Q'(x,t)\|_{L^2_x} &\le \|W(t)(P'(x,t)-Q'(x,t))\|_{L^2_x} \\
&\quad + \left\| \int_0^t W(t-\tau)(M_{P'}-M_{Q'})(x,\tau)\, d\tau \right\|_{L^2_x} \\
&\le c\|P'_0-Q'_0\|_{L^2_x} + c\|(M_{P'}-M_{Q'})(x,t)\|_{X^2_{b-1}}.
\end{aligned}
$$

So, by an argument similar to that above that showed Φ to be a contraction, we have

$$
\|P'(x,t)-Q'(x,t)\|_{L^2_x} \le c\|P'_0-Q'_0\|_{L^2_x} + c(\|P'\|_{X^2_b} + \|Q'\|_{X^2_b})\|P'-Q'\|_{X^2_b}.
$$

Combining this with inequality (25), we dominate the right-hand side by $c\|P_0' - Q_0'\|_{L^2_X}$. This shows

$$\max\left(\|P' - Q'\|_{X^2_b}, \sup_{t\in[0,\frac{1}{2}]} \|P'(x,t) - Q'(x,t)\|_{L^2_X}\right) \le c\|P_0' - Q_0'\|_{L^2_X}, \tag{26}$$

which completes the proof of Theorem 2. □

We now pass to the proof of Theorem 1.

Proof of Theorem 1. Suppose that initial data $P_0 \in L^2_X$ is given. Find $\alpha > 0$ so small that $\alpha^{3/2}\|P_0\|_{L^2_X} < \epsilon_0$, for the ϵ_0 appearing in Theorem 2. Define $P_{0\alpha}(x) = \alpha^2 P_0(\alpha x)$. Then

$$\|P_{0\alpha}\|_{L^2_X} = \alpha^{3/2}\|P_0\|_{L^2_X} < \epsilon_0.$$

So by Theorem 2 we may find $P_\alpha(x,t)$ solving system (16) with initial data $P_{0\alpha}(x)$ for $t \in [0, \frac{1}{2}]$. Consider $P(x,t) = \alpha^{-2}P_\alpha(\alpha^{-1}x, \alpha^{-3}t)$. It is easy to see that P satisfies the first two equations of system (12) since P_α does and $\alpha^{-5}\cdot 0 = 0$. Furthermore, $P(x,0) = \alpha^{-2}P_\alpha(\alpha^{-1}x, 0) = \alpha^{-2}\left[\alpha^2 P_0(\alpha(\alpha^{-1}x))\right] = P_0(x)$. Also $t \in [0, \frac{\alpha^3}{2}]$ if and only if $\alpha^{-3}t \in [0, \frac{1}{2}]$, so $P(x,t)$ is a solution of system (16) with initial data $P_0(x)$ whenever $t \in [0, \frac{\alpha^3}{2}]$. To summarize, given *any* initial L^2_X data P_0, there is a time

$$T_0 = T_0(\|P_0\|_{L^2_X}) = \left(\frac{\epsilon_0}{\|P_0\|_{L^2_X}}\right)^2 \Big/ 2$$

so that we can solve system (16) in $[0, T_0]$. By the conservation equation (2), if $P(x,t)$ is a solution, then $\|P(\cdot,t)\|_{L^2_X}$ is a constant independent of t. Therefore, replacing $t = 0$ by $t = T_0$ and repeating the above argument shows that we can solve system (16) in $[0, 2T_0]$.

The existence of the solution on $[0,\infty)$ now follows from iteration of the above argument.

We now prove uniqueness. This means that if $P(x,t)$ and $P^*(x,t)$ both belong to X^2_b, both are continuous in the sense of condition (4), and both solve system (16) with the same initial data, then they must be equal. Let $\Im = \{t \ge 0 : P(x,t) = P^*(x,t), \text{ for almost all } x\}$. Clearly, $0 \in \Im$. By the continuity condition (4), $\Im$ is a closed set. We now show $\Im$ is also an open set. This will show that $\Im$ is $[0,\infty)$, which gives the uniqueness.

Without loss of generality, we show there exists a neighborhood of 0 in $\Im$. Let $0 \le \alpha, \delta \le 1$ be small positive numbers to be specified later. Define as before,

$$P_\alpha(x,t) = \alpha^2\psi\left(\frac{\alpha^3 t}{\delta}\right)P(\alpha x, \alpha^3 t),$$

$$P^*_\alpha(x,t) = \alpha^2\psi\left(\frac{\alpha^3 t}{\delta}\right)P^*(\alpha x, \alpha^3 t).$$

Simple calculation shows

$$||P_\alpha||^2_{X^2_b} \le c\left\|\psi\left(\frac{t}{\delta}\right)P\right\|^2_{X^2_0} + c\alpha^{6b}\left\|\psi\left(\frac{t}{\delta}\right)P\right\|^2_{X^2_b}. \tag{27}$$

Since P and P^* solves (16) with initial data $P_0(x)$, P_α and P^*_α also solves (16) with initial data $P_{0\alpha}(x) = P_\alpha(x,0) = \alpha^2 P_0(\alpha x)$ for $0 \le t \le \delta$. Use (3.30) of [KPV1] to get $||\psi(\frac{t}{\delta})P||^2_{X^2_0} = ||\psi(\frac{t}{\delta})P||^2_{L^2_X L^2_t} \le c\delta^{1/2}||\psi(\frac{t}{\delta})P||^2_{X^2_b}$. Combine this with Lemma 4; by inequality (27) we have

$$||P_\alpha||^2_{X^2_b} \le c\delta^{\frac{3}{2}-2b}||P||^2_{X^2_b} + c\alpha^{6b}\delta^{1-2b}||P||^2_{X^2_b}.$$

Because $b \in (\frac{1}{2}, \frac{3}{4})$, we can make $||P_\alpha||^2_{X^2_b}$ as small as we like, by first choosing δ small enough to control the first term on the right-hand side and then choosing α small enough to control the second term. Notice that α depends on $||P||_{X^2_b}$. Thus, an appropriate choice of α will ensure that $\|\alpha^2 P_0(\alpha x)\|_{L^2_X} \leq \epsilon_0$, $||P_\alpha||_{X^2_b} \leq r$ and $||P^*_\alpha||_{X^2_b} \leq r$, where ϵ_0, r are the numbers in Theorem 2. By the proof of Theorem 2, we have $P_\alpha(x, t) = P^*_\alpha(x, t)$ for almost all x when $0 \leq t \leq \frac{1}{2}$. Thus, $P(x, t) = P^*(x, t)$ for almost all x when $0 \leq t \leq \min(\frac{\alpha^3}{2}, \delta)$. This shows that $\Im$ contains a neighborhood of 0, which yields the uniqueness.

Let $T > 0$ be given. Find the smallest n such that $nT_0 \geq T$. Let $u_i(t)$ solve system (16) for all $t \in [iT_0, (i+1)T_0]$. The solution $u(t)$ to system (16) for $t \in [0, T]$ coincides with $u_i(t)$ on $[iT_0, (i+1)T_0]$ by the above uniqueness result. Combining the inequality

$$\operatorname*{Essup}_x \int_0^T |u(x,t)|^2\, dt \leq \sum_{i=0}^{n-1} \operatorname*{Essup}_x \int_{iT_0}^{(i+1)T_0} |u(x,t)|^2\, dt \leq \sum_{i=0}^{n-1} \|u_i\|^2_{L^\infty_x L^2_t}$$

with inequality (24) establishes property (5).

This completes the proof. □

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