Name: ______________________ Section: ______________________

Final exam
MATH 217, Fall 2013

Instructions

- The exam is 120 minutes long.
- No calculators or references, including notes, are allowed.
- You must complete the entire exam by yourself. *Do not cheat!*
- Please write in pencil or in blue or black ink.
- You must give full justification for your answers unless otherwise instructed.
- Erase or clearly cross out discarded work; otherwise, it will be considered while grading.
- You may use the backs of pages for additional space or scratch work. Please note where the solution is continued.
- Advice: *Read everything before doing anything!*

Question	Points	Score
1	12	
2	18	
3	9	
4	7	
5	11	
6	10	
7	11	
8	12	
Total:	90	

1. Give the correct definition of each of the following:
   (a) (2 points) The orthogonal complement of a subspace $W \subset \mathbb{R}^n$.

   (b) (2 points) An orthogonal matrix.

   (c) (2 points) A linear map $T\colon V \to W$ of vector spaces being an isomorphism.

   (d) (2 points) The null space of an $m \times n$ matrix A.

   (e) (2 points) A linearly independent subset $\{\vec{v}_1, \dots, \vec{v}_n\}$ of a vector space V (no credit will be given for just "not linearly dependent").

   (f) (2 points) An inner product on a real vector space V.

2. Mark each statement true or false. If it is true, justify it; if it is false, disprove it or give a counterexample.

   (a) (3 points) Suppose that A and B are row-equivalent square matrices. Then they have the same eigenvalues.

   (b) (3 points) There are no unit vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $\vec{u} \cdot \vec{v} = 2$.

   (c) (3 points) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

   (d) (3 points) If A is a diagonalizable $n \times n$ matrix, then every vector in $\mathbb{R}^n$ is an eigenvector of A.

   (e) (3 points) The vectors $\vec{v} = \begin{pmatrix} 3 \\ 1+i \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 6-3i \\ 3+i \end{pmatrix}$ in $\mathbb{C}^2$ are linearly dependent.

   (f) (3 points) A linear transformation is one-to-one if and only if it is onto.

3. Let $$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 3 & 3 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(a) (3 points) Find the set of solutions to $A\vec{x} = \vec{b}$.

(b) (2 points) Find the set of solutions to $A\vec{x} = \vec{0}$.

(c) (4 points) Find the set of least-squares solutions to $A\vec{x} = \vec{b}$.

4. Let $A = \begin{pmatrix} -1 & 2 \\ -16 & 7 \end{pmatrix}$.

   (a) (3 points) Find the eigenvalues of A.

   (b) (4 points) There is some positive real number c such that cA is similar to a rotation matrix through some angle θ. Find c and $\cos\theta$.

5. Let A be the matrix below:

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

(a) (3 points) Find a basis for $\text{Row}(A)$.

(b) (6 points) Find an orthogonal basis for $\text{Row}(A)$.

(c) (2 points) Use your results to find an orthogonal basis for $\text{Nul}(A)^\perp$.

6. Let V be the vector space of all 2×2 matrices. Define, for every $A, B \in V$:

$$\langle A, B \rangle = \operatorname{tr}(A^T B), \qquad \text{where } \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

It is a fact that $(V, \langle , \rangle)$ is an inner product space (which you need not prove). Let A be any 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(a) (3 points) Compute $\|A\|^2 = \langle A, A \rangle$.

(b) (4 points) Let $W \subset V$ be the subspace spanned by the single matrix $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Find the orthogonal projection $\hat{A}$ of A onto W.

(c) (3 points) Write $\hat{A}$ and $A - \hat{A}$ as linear combinations of A and A^T and show that $W^\perp$ is equal to the set Sym of symmetric matrices (those A with $A = A^T$).

7. Let
$$A = \begin{pmatrix} 5 & 0 & 0 \\ 4 & 5 & -4 \\ 4 & 0 & 1 \end{pmatrix}.$$

(a) (3 points) Find the eigenvalues of A.

(b) (8 points) If possible, diagonalize A: find an invertible matrix R and a diagonal matrix D such that $A = RDR^{-1}$, or prove that this is not possible.

8. (12 points) Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ be a set of n vectors. Prove that if the $\vec{v}_i$ are linearly independent in $\mathbb{R}^n$, then the matrices $A_i = \vec{v}_i \vec{v}_i^T$ are linearly independent in the space of $n \times n$ matrices. (Hint: a matrix B is zero if and only if for *every* vector $\vec{x} \in \mathbb{R}^n$, we have $B\vec{x} = \vec{0}$.)