Name: ______________________ Section: ______________________

Final exam
MATH 217, Fall 2013

Instructions

- The exam is 120 minutes long.
- No calculators or references, including notes, are allowed.
- You must complete the entire exam by yourself. *Do not cheat!*
- Please write in pencil or in blue or black ink.
- You must give full justification for your answers unless otherwise instructed.
- Erase or clearly cross out discarded work; otherwise, it will be considered while grading.
- You may use the backs of pages for additional space or scratch work. Please note where the solution is continued.
- Advice: *Read everything before doing anything!*

Question	Points	Score
1	12	
2	18	
3	9	
4	7	
5	11	
6	10	
7	11	
8	12	
Total:	90	

1. Give the correct definition of each of the following:

(a) (2 points) The orthogonal complement of a subspace $W \subset \mathbb{R}^n$.

> **Solution:** The set of all $\vec{v} \in \mathbb{R}^n$ such that for all $\vec{w} \in W$ we have $\vec{v} \cdot \vec{w} = 0$.

(b) (2 points) An orthogonal matrix.

> **Solution:** A square matrix whose columns form an orthonormal set.

(c) (2 points) A linear map $T\colon V \to W$ of vector spaces being an isomorphism.

> **Solution:** A function f that is one-to-one and onto.

(d) (2 points) The null space of an $m \times n$ matrix A.

> **Solution:** The set of all $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{0}$.

(e) (2 points) A linearly independent subset $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of a vector space V (no credit will be given for just "not linearly dependent").

> **Solution:** A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots\}$ such that the only scalars $a_1, a_2, \ldots$ for which
> $$a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots = \vec{0}$$
> are $a_1 = a_2 = \cdots = 0$.

(f) (2 points) An inner product on a real vector space V.

> **Solution:** A function $\langle , \rangle\colon V \times V \to \mathbb{R}$ that is
>
> **Linear:** $\langle v_1 + v_2, w\rangle = \langle v_1, w\rangle + \langle v_2, w\rangle$ and $\langle rv, w\rangle = r\langle v, w\rangle$ for all vectors $v_1, v_2, v, w \in V$ and $r \in \mathbb{R}$;
>
> **Symmetric:** $\langle v, w\rangle = \langle w, v\rangle$ for all vectors $v, w \in V$.
>
> **Positive-definite:** $\langle v, v\rangle \geq 0$ for all vectors $v \in V$, and if $\langle v, v\rangle = 0$ then $v = 0$.

2. Mark each statement true or false. If it is true, justify it; if it is false, disprove it or give a counterexample.

   (a) (3 points) Suppose that A and B are row-equivalent square matrices. Then they have the same eigenvalues.

   Solution: False. For example, the 1×1 matrices (1) and (2) are row-equivalent but have eigenvalues 1 and 2, respectively.

   (b) (3 points) There are no unit vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ such that $\vec{u} \cdot \vec{v} = 2$.

   Solution: True; this violates the Cauchy–Schwarz inequality:

   $$2 = \vec{u} \cdot \vec{v} \not\leq \|\vec{u}\|\|\vec{v}\| = 1 \cdot 1 = 1.$$

   (c) (3 points) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

   Solution: False; for example, the identity matrix I_n has only the eigenvalue 1 but is diagonal.

   (d) (3 points) If A is a diagonalizable $n \times n$ matrix, then every vector in $\mathbb{R}^n$ is an eigenvector of A.

   Solution: False; the matrix $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ is diagonal but $(1, 1)$ is not an eigenvector since $A(1, 1) = (1, 2)$.

   (e) (3 points) The vectors $\vec{v} = \begin{pmatrix} 3 \\ 1+i \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 6-3i \\ 3+i \end{pmatrix}$ in $\mathbb{C}^2$ are linearly dependent.

   Solution: True; we have $\vec{w} = (2 - i)\vec{v}$.

   (f) (3 points) A linear transformation is one-to-one if and only if it is onto.

   Solution: False; only for square matrices, otherwise $\begin{pmatrix} 1 & 1 \end{pmatrix}$ is a counterexample (onto but not one-to-one).

3. Let $$A = \begin{pmatrix} 2 & 1 & 1 \\ 6 & 3 & 3 \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(a) (3 points) Find the set of solutions to $A\vec{x} = \vec{b}$.

Solution: We row-reduce the augmented matrix:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 6 & 3 & 3 & 3 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

The equation is inconsistent, so the solution set is empty.

(b) (2 points) Find the set of solutions to $A\vec{x} = \vec{0}$.

Solution: Having already found the echelon form of the coefficient matrix, we get its kernel

$$\text{span}\left\{ \begin{pmatrix} 1/2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} \right\}.$$

(c) (4 points) Find the set of least-squares solutions to $A\vec{x} = \vec{b}$.

Solution: We need to compute:

$$A^T A = \begin{pmatrix} 40 & 20 & 20 \\ 20 & 10 & 10 \\ 20 & 10 & 10 \end{pmatrix} \qquad A^T\vec{b} = \begin{pmatrix} 22 \\ 11 \\ 11 \end{pmatrix}.$$

Then we may solve the (consistent) system $A^T A\hat{x} = A^T\vec{b}$, whose reduced augmented matrix is

$$\left(\begin{array}{ccc|c} 40 & 20 & 20 & 22 \\ 20 & 10 & 10 & 11 \\ 20 & 10 & 10 & 11 \end{array}\right) \xrightarrow{\text{row reduce}} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{1}{2} & \frac{11}{20} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The set of all least-squares solutions is therefore

$$\left\{ \begin{pmatrix} 11/20 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 1/2 \\ -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1/2 \\ 0 \\ -1 \end{pmatrix} \middle| \ s, t \in \mathbb{R} \right\}.$$

4. Let $A = \begin{pmatrix} -1 & 2 \\ -16 & 7 \end{pmatrix}$.

   (a) (3 points) Find the eigenvalues of A.

   Solution: We have $\operatorname{tr} A = 6$ and $\det A = 25$, so the characteristic polynomial is

   $$p_A(\lambda) = \lambda^2 - 6\lambda + 25,$$

   whose roots (by the quadratic formula) are $3 \pm 4i$.

   (b) (4 points) There is some positive real number c such that cA is similar to a rotation matrix through some angle θ. Find c and $\cos\theta$.

   Solution: c^{-1} is the length of (either) eigenvalue, namely 5, and $\cos\theta$ is the cosine of the angle of (either) eigenvalue, namely 3/5.

5. Let A be the matrix below:

$$A = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 3 \\ 1 & 0 & 1 & 2 \end{pmatrix}$$

(a) (3 points) Find a basis for Row(A).

Solution: For ease of notation, we use the fact that $\mathrm{Row}(A)^T = \mathrm{Col}(A^T)$. Performing row-reduction on

$$A^T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 2 \end{pmatrix}$$

we find that its first three columns span its column space, so the desired basis is

$$\mathscr{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}^T \right\}.$$

(b) (6 points) Find an orthogonal basis for Row(A).

Solution: We need to apply the Gram–Schmidt process to $\mathscr{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, computed above, obtaining an orthogonal basis $\mathscr{O} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$. We begin by setting $\vec{u}_1 = \vec{v}_1$, and finding

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}^T - \frac{2}{6}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}^T = \begin{pmatrix} -1/3 \\ 1 \\ -1/3 \\ 1/3 \end{pmatrix}^T \to \begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \end{pmatrix}^T.$$

The last orthogonal vector is

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 - \frac{\vec{v}_3 \cdot \vec{u}_2}{\vec{u}_2 \cdot \vec{u}_2}\vec{u}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 3 \end{pmatrix}^T - \frac{7}{6}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix}^T - \frac{-8}{12}\begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \end{pmatrix}^T = \begin{pmatrix} -1/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}^T \to \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}^T.$$

(c) (2 points) Use your results to find an orthogonal basis for $\mathrm{Nul}(A)^{\perp}$.

Solution: Since $\mathrm{Nul}(A)^{\perp} = \mathrm{Col}(A^T) = \mathrm{Row}(A)^T$, the basis computed previously works (without transposing).

6. Let V be the vector space of all 2×2 matrices. Define, for every $A, B \in V$:

$$\langle A, B \rangle = \operatorname{tr}(A^T B), \qquad \text{where } \operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

It is a fact that $(V, \langle , \rangle)$ is an inner product space (which you need not prove). Let A be any 2×2 matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(a) (3 points) Compute $\|A\|^2 = \langle A, A \rangle$.

Solution: We have

$$A^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix}$$

whose trace is $a^2 + b^2 + c^2 + d^2$.

(b) (4 points) Let $W \subset V$ be the subspace spanned by the single matrix $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Find the orthogonal projection $\hat{A}$ of A onto W.

Solution: First, we have $\|S\|^2 = 0^2 + 1^2 + (-1)^2 + 0^2 = 2$. Second, we have

$$\langle A, S \rangle = \operatorname{tr}\left[\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] = \operatorname{tr}\begin{pmatrix} -c & a \\ -d & b \end{pmatrix} = b - c.$$

Then the projection formula is

$$\hat{A} = \operatorname{proj}_W(A) = \frac{\langle A, S \rangle}{\langle S, S \rangle} S = \frac{b - c}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) (3 points) Write $\hat{A}$ and $A - \hat{A}$ as linear combinations of A and A^T and show that $W^\perp$ is equal to the set Sym of symmetric matrices (those A with $A = A^T$).

Solution: We have $\hat{A} = (A - A^T)/2$ and, thus $A - \hat{A} = (A + A^T)/2$. We have $\hat{A} = 0$ if and only if $b = c$, which is equivalent to $A = A^T$.

7. Let

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 4 & 5 & -4 \\ 4 & 0 & 1 \end{pmatrix}.$$

(a) (3 points) Find the eigenvalues of A.

Solution: Expanding down the 2nd column, get the char. polynomial in the form

$$p(\lambda) = (5-\lambda)(\lambda^2 - 6\lambda + 5).$$

Obviously $\lambda_1 = 5$ (and hopefully they will see that $\hat{e}_2$ is an associated eigenvector). The roots of the quadratic factor are 5 and 1. So 5 has algebraic multiplicity two.

(b) (8 points) If possible, diagonalize A: find an invertible matrix R and a diagonal matrix D such that $A = RDR^{-1}$, or prove that this is not possible.

Solution: For $\lambda = 5$: Must find a basis of the null space of

$$\begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & -4 \\ 4 & 0 & -4 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This leads to the eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

The geometric multiplicity is two, so already we can say diagonalization is possible.
For $\lambda = 1$: Must find a basis of the null space of

$$\begin{pmatrix} 4 & 0 & 0 \\ 4 & 4 & -4 \\ 4 & 0 & 0 \end{pmatrix} \xrightarrow{\text{row reduce}} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This leads to the eigenvector

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

So, if

$$R = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

then

$$A = RDR^{-1},$$

by “general theory".

8. (12 points) Let $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ be a set of n vectors. Prove that if the $\vec{v}_i$ are linearly independent in $\mathbb{R}^n$, then the matrices $A_i = \vec{v}_i \vec{v}_i^T$ are linearly independent in the space of $n \times n$ matrices. (Hint: a matrix B is zero if and only if for *every* vector $\vec{x} \in \mathbb{R}^n$, we have $B\vec{x} = \vec{0}$.)

Solution: Suppose the $\vec{v}_i$ are linearly independent, and consider a linear relation

$$\sum_{i=1}^{n} a_i A_i = 0.$$

Both sides are $n \times n$ matrices, so we may multiply by any vector $\vec{x} \in \mathbb{R}^n$, giving

$$\sum_{i=1}^{n} a_i A_i \vec{x} = \sum_{i=1}^{n} a_i (\vec{v}_i^T \vec{x}) \vec{v}_i = \vec{0}.$$

By hypothesis, then, all coefficients $a_i \vec{v}_i^T \vec{x} = 0$. Write this as

$$a_i (\vec{v}_i \cdot \vec{x}) = 0.$$

Taking $\vec{x} = \vec{v}_i$, we find that $a_i \|\vec{v}_i\|^2 = 0$, and since $\vec{v}_i \neq \vec{0}$, we get $a_i = 0$, so the relation is trivial; thus the A_i are linearly independent.