Math 217 – Midterm

Spring 2014

Time: 120 mins.

1. Answer each question in the space provided.
2. Clearly explain and justify your reasoning at each step.
3. No calculators, notes, or other outside assistance allowed.

Name: ______________________________ Section: ______________

Question	Points	Score
1	10	
2	15	
3	12	
4	18	
5	11	
6	12	
7	12	
8	10	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

(a) (2 points) The *image* of an n x m matrix A.

(b) (2 points) V is a *subspace* of $\mathbb{R}^n$.

(c) (2 points) A *linear relation* between vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$.

(d) (2 points) A *basis* of a subspace V of $\mathbb{R}^n$.

(e) (2 points) The *dimension* of a subspace V of $\mathbb{R}^n$.

2. State whether each statement is True or False and justify your answer.

(a) (3 points) There exists a 3 x 4 matrix A of rank 3 such that $A \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \mathbf{0}$.

(b) (3 points) If A and B are 3×2 matrices of rank 2, then $\text{rref}(A) = \text{rref}(B)$.

(c) (3 points) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent vectors in $\mathbb{R}^n$, then all three vectors must be parallel to each other.

(d) (3 points) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\text{im}(T)$ is a plane and $\ker(T)$ is a plane.

(e) (3 points) If $A \in \mathbb{R}^{n\times n}$ and $\text{rank}(A) = n$, then $A = I_n$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \qquad \text{and} \qquad T\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}.$$

(a) (6 points) Can you tell for certain what $T\begin{bmatrix}3\\4\\0\end{bmatrix}$ is? If yes, find it. If not, why not?

(b) (6 points) Can you tell for certain what $T\begin{bmatrix}1\\2\\3\end{bmatrix}$ is? If yes, find it. If not, why not?

4. Let

$$A = \begin{bmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x}$.

(a) (8 points) Show that $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of $\mathbb{R}^3$.

(b) (10 points) Find $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix for T.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the vector $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

If you should need the formula, $T(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$ *for all* $\mathbf{x} \in \mathbb{R}^3$*. It is possible to solve this problem without finding the matrix for* T*.*

(a) (4 points) Find a basis for $\mathrm{im}(T)$.

(b) (7 points) Find a basis for $\ker(T)$.

6. (12 points) Consider P_2, the linear space of all polynomials of degree ≤ 2, and a subspace V of P_2 defined as

$$V = \left\{ f \in P_2 : \int_0^1 f(t)\, dt = 0 \right\}.$$

Find a basis for V. What is the dimension of V?

7. (a) (6 points) Let $A \in \mathbb{R}^{n \times n}$ such that $A^2 = 0$. Prove that $\text{im}(A) \subseteq \ker(A)$.

(b) (6 points) Let $A \in \mathbb{R}^{2 \times 2}$ such that $A^2 = 0$ but $A \neq 0$. Prove that $\text{im}(A) = \ker(A)$.
Hint : Use part (a).

8. (10 points) Let $A, B \in \mathbb{R}^{n\times m}$.

   Prove that $\operatorname{im}(A) \subseteq \operatorname{im}(B)$ if and only if there exists $C \in \mathbb{R}^{m\times m}$ such that $A = BC$.