Math 217 – Midterm

Spring 2014

Time: 120 mins.

1. Answer each question in the space provided.
2. Clearly explain and justify your reasoning at each step.
3. No calculators, notes, or other outside assistance allowed.

Name: ______________________ Section: __________

Question	Points	Score
1	10	
2	15	
3	12	
4	18	
5	11	
6	12	
7	12	
8	10	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (2 points) The *image* of an n x m matrix A.

   > **Solution:** The *image* of an n x m matrix A is $\{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^m\}$.

   (b) (2 points) V is a *subspace* of $\mathbb{R}^n$.

   > **Solution:** V is a *subspace* of $\mathbb{R}^n$ if $V \subseteq \mathbb{R}^n$ and V is closed under linear combinations. This means that $\mathbf{0} \in V$, and for every $\mathbf{v}$ and $\mathbf{w} \in V$, $\mathbf{v} + \mathbf{w} \in V$ and $k\mathbf{v} \in V$ for any scalar k.

   (c) (2 points) A *linear relation* between vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$.

   > **Solution:** A *linear relation* between vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ is an equation of the form
   > $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_m = \mathbf{0}$$
   > for some scalars $c_1, c_2, \ldots, c_m$.

   (d) (2 points) A *basis* of a subspace V of $\mathbb{R}^n$.

   > **Solution:** A *basis* of a subspace V of $\mathbb{R}^n$ is a list of linearly independent vectors whose span is V.

   (e) (2 points) The *dimension* of a subspace V of $\mathbb{R}^n$.

   > **Solution:** The *dimension* of a subspace V of $\mathbb{R}^n$ is the number of vectors in a basis for V.

2. State whether each statement is True or False and justify your answer.

(a) (3 points) There exists a 3 x 4 matrix A of rank 3 such that $A\begin{bmatrix}1\\1\\1\\-1\end{bmatrix} = \mathbf{0}$.

Solution: True. For example, let

$$A = \begin{bmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}.$$

(b) (3 points) If A and B are 3×2 matrices of rank 2, then $\text{rref}(A) = \text{rref}(B)$.

Solution: True. $\text{rref}(A)$ and $\text{rref}(B)$ must have leading ones in the first two rows. Since there must be zeroes above and below any leading ones, the only possibility is $\text{rref}(A) = \text{rref}(B) = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$.

(c) (3 points) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent vectors in $\mathbb{R}^n$, then all three vectors must be parallel to each other.

Solution: False. For example let $\mathbf{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix}1\\1\\0\end{bmatrix}$. These vectors all lie on a single plane, but not a single line.

(d) (3 points) There exists a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\text{im}(T)$ is a plane and $\ker(T)$ is a plane.

Solution: False. The dimension of a plane is 2, so if $\text{im}(T)$ and $\ker(T)$ are both planes we have $\dim(\text{im}(T)) + \dim(\ker(T)) = 2 + 2 = 4$. This is a violation of the rank-nullity theorem, which says that we must have $\dim(\text{im}(T)) + \dim(\ker(T)) = 3$.

(e) (3 points) If $A \in \mathbb{R}^{n \times n}$ and $\text{rank}(A) = n$, then $A = I_n$.

Solution: False. We could have, for example, $A = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} \qquad \text{and} \qquad T\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}5\\2\end{bmatrix}.$$

(a) (6 points) Can you tell for certain what $T\begin{bmatrix}3\\4\\0\end{bmatrix}$ is? If yes, find it. If not, why not?

Solution: Yes. Notice that $\begin{bmatrix}3\\4\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\\0\end{bmatrix}$. Since T is a linear transformation, we have

$$T\begin{bmatrix}3\\4\\0\end{bmatrix} = T\left(\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}1\\2\\0\end{bmatrix}\right) = T\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\,T\begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}5\\2\end{bmatrix} = \begin{bmatrix}11\\6\end{bmatrix}.$$

(b) (6 points) Can you tell for certain what $T\begin{bmatrix}1\\2\\3\end{bmatrix}$ is? If yes, find it. If not, why not?

Solution: No. The vector $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\\0\end{bmatrix}$. This is clear, since the last component of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is nonzero, whereas the last components of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\\0\end{bmatrix}$ are both zero.

4. Let

$$A = \begin{bmatrix} 4 & 1 & 1 \\ -5 & 0 & -3 \\ -1 & -1 & 2 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x}$.

(a) (8 points) Show that $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of $\mathbb{R}^3$.

Solution: Let $S = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} \in \mathbb{R}^{3\times 3}$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if $\operatorname{rank}(S) = 3$. Using Gauss–Jordan elimination, we find

$$S \sim \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -1/2 \end{bmatrix},$$

where $\sim$ indicates row equivalence. It is therefore clear that $\operatorname{rank}(S) = 3$, and so that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Since $\dim(\mathbb{R}^3) = 3$, the three linearly independent vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ must span $\mathbb{R}^3$, and therefore form a basis for $\mathbb{R}^3$.

(b) (10 points) Find $[T]_{\mathcal{B}}$, the $\mathcal{B}$-matrix for T.

Solution: We can find the $\mathcal{B}$-matrix for T column by column. The first, second, and third columns of $[T]_{\mathcal{B}}$ are, respectively, $[A\mathbf{v}_1]_{\mathcal{B}}$, $[A\mathbf{v}_2]_{\mathcal{B}}$, and $[A\mathbf{v}_3]_{\mathcal{B}}$. The vectors $A\mathbf{v}_1$, $A\mathbf{v}_2$, and $A\mathbf{v}_3$ are the first, second, and third columns, respectively, of the matrix AS, where S is as in part (a). Matrix multiplication gives

$$AS = \begin{bmatrix} 0 & -2 & -1 \\ -3 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

The columns of this matrix are

$$A\mathbf{v}_1 = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} = 3\mathbf{v}_1, \quad A\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} = 2\mathbf{v}_2, \quad A\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = 1\mathbf{v}_3.$$

It follows that

$$[A\mathbf{v}_1]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \quad [A\mathbf{v}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \quad [A\mathbf{v}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and so

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto the vector $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$.

If you should need the formula, $T(\mathbf{x}) = \left(\dfrac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$ *for all* $\mathbf{x} \in \mathbb{R}^3$. *It is possible to solve this problem without finding the matrix for* T.

(a) (4 points) Find a basis for $\operatorname{im}(T)$.

Solution: Since each vector in $\operatorname{im}(T)$ is a scalar multiple of $\mathbf{w}$, the vector $\mathbf{w}$ forms a basis for $\operatorname{im}(T)$.

(b) (7 points) Find a basis for $\ker(T)$.

Solution: A vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in $\ker(T)$ if and only if $\mathbf{x} \cdot \mathbf{w} = 0$. This implies the equation $x_1 + 2x_2 - 2x_3 = 0$, or equaivalently $x_1 = -2x_2 + 2x_3$. Letting x_2 and x_3 be free parameters, the solution set to this equation is

$$\left\{ x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; x_2, x_3 \in \mathbb{R} \right\}.$$

The vectors $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ are clearly linearly independent, and so they form a basis for $\ker(T)$.

6. (12 points) Consider P_2, the linear space of all polynomials of degree ≤ 2, and a subspace V of P_2 defined as

$$V = \left\{ f \in P_2 : \int\limits_0^1 f(t)\,dt = 0 \right\}.$$

Find a basis for V. What is the dimension of V?

Solution: Let $f(t) = at^2 + bt + c \in P_2$ for some $a, b, c \in \mathbb{R}$. Then $f \in V$ if and only if

$$\int_0^1 (at^2 + bt + c)\,dt = \frac{a}{3} + \frac{b}{2} + c = 0,$$

which holds if $c = -\frac{a}{3} - \frac{b}{2}$. Thus $f \in V$ if and only if f is of the form

$$f(t) = at^2 + bt - \frac{a}{3} - \frac{b}{2},$$

and so

$$V = \left\{ at^2 + bt - \frac{a}{3} - \frac{b}{2} \,\middle|\, a, b \in \mathbb{R} \right\} = \left\{ a\left(t^2 - \frac{1}{3}\right) + b\left(t - \frac{1}{2}\right) \,\middle|\, a, b \in \mathbb{R} \right\}.$$

We see then that V is spanned by the two polynomials $t^2 - \frac{1}{3}$ and $t - \frac{1}{2}$. These polynomials are clearly linearly independent (since they have leading terms of different orders), and so they form a basis for V. It follows that $\dim(V) = 2$.

7. (a) (6 points) Let $A \in \mathbb{R}^{n\times n}$ such that $A^2 = 0$. Prove that $\operatorname{im}(A) \subseteq \ker(A)$.

Solution: Let $\mathbf{y} \in \operatorname{im}(A)$. Then by definition, $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Thus $A\mathbf{y} = A(A\mathbf{x}) = A^2\mathbf{x}$. Since we assumed that $A^2 = 0$, we thus have $A\mathbf{y} = \mathbf{0}$, and so $\mathbf{y} \in \ker(A)$ by definition. This proves that every element of $\operatorname{im}(A)$ is also in $\ker(A)$, and so $\operatorname{im}(A) \subseteq \ker(A)$.

(b) (6 points) Let $A \in \mathbb{R}^{2\times 2}$ such that $A^2 = 0$ but $A \neq 0$. Prove that $\operatorname{im}(A) = \ker(A)$.
Hint : Use part (a).

Solution: If $A \neq 0$, then $\operatorname{rank}(A) \neq 0$, and so $\operatorname{rank}(A) = 1$ or $\operatorname{rank}(A) = 2$. If $\operatorname{rank}(A) = 2$, then $\operatorname{im}(A) = \mathbb{R}^2$ and $\ker(A) = \{\mathbf{0}\}$, which contradicts the result of part (a), $\operatorname{im}(A) \subseteq \ker(A)$. The only possibility then is $\operatorname{rank}(A) = 1$, and the rank-nullity theorem implies that $\dim(\ker(A)) = 1$ as well. Thus $\operatorname{im}(A)$ and $\ker(A)$ are both lines. By part (a), $\operatorname{im}(A) \subseteq \ker(A)$, but the only way a line can be a subset of another line is if they are in fact the same line. Thus $\operatorname{im}(A) = \ker(A)$.

8. (10 points) Let $A, B \in \mathbb{R}^{n\times m}$.

Prove that $\operatorname{im}(A) \subseteq \operatorname{im}(B)$ if and only if there exists $C \in \mathbb{R}^{m\times m}$ such that $A = BC$.

Solution: The proof of this if-and-only-if statement consists of two parts.

First assume that there is a matrix $C \in \mathbb{R}^{m\times m}$ such that $A = BC$. Let $\mathbf{y} \in \operatorname{im}(A) \subset \mathbb{R}^n$. Then by definition, $\mathbf{y} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^m$. Since $A = BC$, we have $\mathbf{y} = (BC)\mathbf{x} = B(C\mathbf{x})$, where $C\mathbf{x} \in \mathbb{R}^m$. This shows each element of $\operatorname{im}(A)$ is also an element of $\operatorname{im}(B)$, and so $\operatorname{im}(A) \subseteq \operatorname{im}(B)$.

Now assume that $\operatorname{im}(A) \subseteq \operatorname{im}(B)$. Let $\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_m \in \mathbb{R}^n$ denote the column vectors of A, and $\mathbf{w}_1, \mathbf{w}_2, \cdots \mathbf{w}_m \in \mathbb{R}^n$ denote the column vectors of B. If $\operatorname{im}(A) \subseteq \operatorname{im}(B)$, then in particular each column vector of A, $\mathbf{v}_j$, is a linear combination of the column vectors of B, i.e., for each $j = 1, 2, \ldots, m$,

$$\mathbf{v}_j = c_{1j}\mathbf{w}_1 + c_{2j}\mathbf{w}_2 + \cdots + c_{mj}\mathbf{w}_m,$$

for some scalars $c_{1j}, c_{2j}, \ldots, c_{mj}$. This equation can be written in the matrix form

$$\mathbf{v}_j = B\begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}.$$

Combining this equation for all columns $j = 1, 2, \ldots, m$, we obtain the matrix equation $A = BC$, where

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1m} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & c_{m3} & \cdots & c_{mm} \end{bmatrix}.$$