Math 217 – Final Exam

Winter 2014

Time: 120 mins.

1. Answer each question in the space provided.
2. Use only the paper provided with this exam.
3. Remember to show all your work.
4. No calculators, notes, or other outside assistance allowed.

Name: ______________________________ Section: ____________

Question	Points	Score
1	8	
2	15	
3	14	
4	14	
5	12	
6	15	
7	12	
8	10	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (2 points) An *eigenvector* of an $n \times n$ matrix A.

   (b) (2 points) The *kernel* of an $n \times m$ matrix A.

   (c) (2 points) The *dimension* of a subspace V of $\mathbb{R}^n$.

   (d) (2 points) The *geometric multiplicity* of an eigenvalue λ of an $n \times n$ matrix A.

2. State whether each statement is True or False and justify your answer(e.g., by giving a short proof or counterexample).

   (a) (3 points) Every diagonalizable $n \times n$ matrix is invertible.

   (b) (3 points) The matrix A of an orthogonal projection onto any subspace V of $\mathbb{R}^n$ is diagonalizable.

   (c) (3 points) If $A, B \in \mathbb{R}^{2\times 2}$ have the same trace and the same determinant as each other, then A and B have the same eigenvalues.

   (d) (3 points) If $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times n}$, then $\det(AB) = \det(BA)$.

   (e) (3 points) Any $n \times n$ matrix with rank n is similar to I_n.

3. Let

$$A = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 1 & 2 & 1 & -2 \\ 1 & 1 & 2 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

(a) (7 points) 1 is an eigenvalue of A. Find a basis of the eigenspace E_1.

(b) (7 points) Diagonalize A.

4. Let $\mathbf{v}_1 = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} b \\ c \\ c \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$, where a, b, c, d, e and f are all positive real constants.

   (a) (4 points) What restrictions, if any, must be placed on a-f for these vectors to form a basis for $\mathbb{R}^3$?

   (b) (6 points) For your choice of values from (a), find an orthonormal basis $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ for $\mathbb{R}^3$ derived from $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

   (c) (4 points) Find the QR factorization of $M = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$.

5. Prove the following statements.

(a) (6 points) If $A \in \mathbb{R}^{n\times n}$ is symmetric and $\mathbf{v} \in \operatorname{im}(A), \mathbf{w} \in \ker(A)$, then $\mathbf{v}$ is orthogonal to $\mathbf{w}$.

(b) (6 points) If k is a positive odd integer and $B \in \mathbb{R}^{n\times n}$ is a symmetric matrix, then there exists a symmetric matrix $C \in \mathbb{R}^{n\times n}$ such that $B = C^k$.

6. Consider the linear transformation $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ such that $T(A) = A - A^T$.

   (a) (5 points) Find a basis for $\ker(T)$. Describe this subspace.

   (b) (5 points) Find a basis for $\operatorname{im}(T)$. Describe this subspace.

   (c) (5 points) Prove that T is diagonalizable. Find an eigenbasis $\mathcal{B}$ for T and find the $\mathcal{B}$-matrix $[T]_{\mathcal{B}}$.

7. Let $A = \begin{bmatrix} 2 & b \\ -1 & 0 \end{bmatrix}$.

   (a) (6 points) For what values of b is A diagonalizable over $\mathbb{R}$? Over $\mathbb{C}$? For what values of b is A not diagonalizable over $\mathbb{R}$ or $\mathbb{C}$?

   (b) (6 points) If $b = 5$, find a rotation-scaling matrix B similar to A, and the similarity matrix S. What are the scaling factor and rotation angle of B?

8. (10 points) Prove that if V and W are subspaces of $\mathbb{R}^n$ such that $V \cap W = \{\mathbf{0}\}$, then $\dim(V) + \dim(W) \leq n$.