Math 217 – Midterm 1

Winter 2014

Time: 120 mins.

1. Answer each question in the space provided.
2. If you run out of space for an answer, continue on the back of the page.
3. Remember to show all your work.
4. No calculators, notes, or other outside assistance allowed.

Name: ______________________________ Section: ____________

Question	Points	Score
1	8	
2	15	
3	20	
4	15	
5	10	
6	18	
7	6	
8	8	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (2 points) An *invertible* function $f : X \to Y$.

   (b) (2 points) The *kernel* of an n x m matrix.

   (c) (2 points) The *span* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$.

   (d) (2 points) A *basis* of a linear subspace V of $\mathbb{R}^n$.

2. State whether each statement is True or False and justify your answer.

   (a) (3 points) If $\text{rref}(A) = \text{rref}(B)$, then $\text{im}(A) = \text{im}(B)$.

   (b) (3 points) If V and W are linear subspaces of $\mathbb{R}^n$, then $V \cup W$ is also a linear subspace of $\mathbb{R}^n$.

   (c) (3 points) If A and B are n x n matrices such that $\ker(A) = \{\mathbf{0}\}$ and $\ker(B) = \{\mathbf{0}\}$, then $\ker(AB) = \{\mathbf{0}\}$.

   (d) (3 points) There exists an invertible n x n matrix with exactly $n - 1$ non-zero entries.

   (e) (3 points) If A and B are n x n matrices such that $AB = B$, then $A = I$.

3. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation with the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

(a) (8 points) Find a basis for $\ker(T)$.

(b) (3 points) Find a basis for $\operatorname{im}(T)$.

(c) (3 points) What is $\operatorname{rank}(A)$? How is $\operatorname{rank}(A)$ related to $\dim(\ker(A))$?

(d) (3 points) Is T one-to-one? Justify your answer.

(e) (3 points) Is T onto? Justify your answer.

4. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation by 60° counterclockwise. Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the reflection about the line $y = x$.

   (a) (9 points) Find the standard matrix of the linear transformation $T \circ S$.

   (b) (6 points) Find the standard matrix of the linear transformation T^9.

5. (10 points) Let V and W be linear subspaces of $\mathbb{R}^n$. Define

$$V + W = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V, \mathbf{w} \in W\}.$$

Prove that $V + W$ is a linear subspace of $\mathbb{R}^n$.

6.

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x}$. $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of $\mathbb{R}^3$.

(a) (12 points) Find $B = [T]_{\mathcal{B}}$.

(b) (6 points) Prove that $[T^2]_{\mathcal{B}} = B^2$.

7. (6 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the invertible linear transformation such that

$$T\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Find the standard matrix for T^{-1}.

8. (8 points) Prove that if A and B are n x n matrices and AB is invertible, then A and B are invertible.