Math 217 – Midterm 1

Winter 2014

Solutions

Name: ______________________ Section: __________

Question	Points	Score
1	8	
2	15	
3	20	
4	15	
5	10	
6	18	
7	6	
8	8	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (2 points) An *invertible* function $f : X \to Y$.
   Solution:
   A function f is invertible if $\forall y \in Y$, $\exists$ a unique $x \in X$ s.t. $f(x) = y$.

   (b) (2 points) The *kernel* of an n x m matrix.
   Solution:
   Let A be this matrix. The kernel of A is $\ker(A) = \{\mathbf{x} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{0}\}$.

   (c) (2 points) The *span* of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m \in \mathbb{R}^n$.
   Solution:
   $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = \{\mathbf{x} : \mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m, \text{ for some } c_1, \ldots, c_m \in \mathbb{R}\}$

   (d) (2 points) A *basis* of a linear subspace V of $\mathbb{R}^n$.
   Solution:
   A basis of V is a set of linearly independent vectors that span V.

2. State whether each statement is True or False and justify your answer.

(a) (3 points) If $\text{rref}(A) = \text{rref}(B)$, then $\text{im}(A) = \text{im}(B)$.

Solution: FALSE. Let $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $\text{rref}(A) = \text{rref}(B) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, however $\text{im}(A)$ is the x-axis while $\text{im}(B)$ is the y-axis.

(b) (3 points) If V and W are linear subspaces of $\mathbb{R}^n$, then $V \cup W$ is also a linear subspace of $\mathbb{R}^n$.

Solution: FALSE. Let $V = \text{span}(\mathbf{e}_1)$ and $W = \text{span}(\mathbf{e}_2)$ in $\mathbb{R}^2$. V and W are linear subspaces of $\mathbb{R}^2$. If $V \cup W$ is also a linear subspace, it must be closed under addition. $\mathbf{e}_1, \mathbf{e}_2 \in V \cup W$, but $\mathbf{e}_1 + \mathbf{e}_2 \notin V \cup W$.

(c) (3 points) If A and B are n x n matrices such that $\ker(A) = \{\mathbf{0}\}$ and $\ker(B) = \{\mathbf{0}\}$, then $\ker(AB) = \{\mathbf{0}\}$.

Solution: TRUE. A square matrix is invertible if and only if its kernel is $\{\mathbf{0}\}$. $\ker(A) = \{\mathbf{0}\}$ and $\ker(B) = \{\mathbf{0}\}$ implies A and B are invertible. If A, B are invertible, then their product AB is also invertible, which implies $\ker(AB) = \{\mathbf{0}\}$.

(d) (3 points) There exists an invertible n x n matrix with exactly $n - 1$ non-zero entries.

Solution: FALSE. An n x n matrix A is invertible if and only if $\text{rank}(A) = n$. If A has exactly $n - 1$ non-zero entries, then it must have at least one row of 0s. Then $\text{rank}(A) < n$ and it cannot be invertible.

(e) (3 points) If A and B are n x n matrices such that $AB = B$, then $A = I$.

Solution: FALSE. Let A be any 2 x 2 matrix and let $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = B$.

3. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be a linear transformation with the standard matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

(a) (8 points) Find a basis for $\ker(T)$.
Solution: We first find $\text{rref}(A)$.

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -2/3 & 1/3 \\ 0 & 1 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 2/3 & 1/3 \end{bmatrix}$$

Thus the fourth and fifth columns are redundant, and we have the basis

$$\text{basis of } \ker(T) = \left\{ \begin{pmatrix} -2 \\ 1 \\ 2 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \\ 0 \\ -3 \end{pmatrix} \right\}.$$

(b) (3 points) Find a basis for $\text{im}(T)$.
Solution: A basis for $\text{im}(T)$ is provided by the columns in A corresponding to the pivots in $\text{rref}(A)$:

$$\text{basis of } \text{im}(T) = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}.$$

Though we note that in this case because $\text{rank}(A) = 3 = \dim(\mathbb{R}^3)$, the image is all of $\mathbb{R}^3$, so any three linear independent vectors in $\mathbb{R}^3$ will be a basis of the image.

(c) (3 points) What is $\text{rank}(A)$? How is $\text{rank}(A)$ related to $\dim(\ker(A))$?
Solution: There are three vectors in our basis for the image, so

$$\text{rank}(A) = 3 = (\text{number of columns in } A) - \dim(\ker(A))$$

(by the rank-nullity theorem).

(d) (3 points) Is T one-to-one? Justify your answer.
Solution: No. $\ker(A) \neq \{\mathbf{0}\}$, so it is clearly not one-to-one.

(e) (3 points) Is T onto? Justify your answer.
Solution: Yes. $\dim(\text{im}(A)) = 3 = \dim(\mathbb{R}^3)$, so the image is all of the target space.

4. Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the rotation by 60° counterclockwise. Let $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the reflection about the line $y = x$.

   (a) (9 points) Find the standard matrix of the linear transformation $T \circ S$.
   Solution:

$$A_T = \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix} \implies A_S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A_{(T\circ S)} = A_T A_S = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{3}/2 & 1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

   (b) (6 points) Find the standard matrix of the linear transformation T^9.
   Solution: T^9 is rotation by $(60)(9)^\circ = 540^\circ$ counterclockwise, effectively rotation by 180° counterclockwise or reflection across the origin.

$$T^9\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix} \implies A_{T^9} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

5. (10 points) Let V and W be linear subspaces of $\mathbb{R}^n$. Define

$$V + W = \{\mathbf{v} + \mathbf{w} : \mathbf{v} \in V, \mathbf{w} \in W\}.$$

Prove that $V + W$ is a linear subspace of $\mathbb{R}^n$.

Solution :

In order to check that $V+W$ is a subspace we must check three properties: (1) $\mathbf{0} \in V+W$, (2) if $\mathbf{u}', \mathbf{u}'' \in V+W$ then $\mathbf{u}'+\mathbf{u}'' \in V+W$, and (3) if $\mathbf{u} \in V$ and $\lambda \in \mathbb{R}$ then $\lambda\mathbf{u} \in V+W$.

1. Since V and W are subspaces of $\mathbb{R}^n$, $\mathbf{0} \in V$ and $\mathbf{0} \in W$. Therefore, by construction $\mathbf{0} = \mathbf{0} + \mathbf{0} \in V + W$.
2. Let $\mathbf{u}', \mathbf{u}'' \in V + W$. Then there are vectors $\mathbf{v}', \mathbf{v}'' \in V$ and $\mathbf{w}', \mathbf{w}'' \in W$ such that $\mathbf{u}' = \mathbf{v}' + \mathbf{w}'$ and $\mathbf{u}'' = \mathbf{v}'' + \mathbf{w}''$. Now,

$$\mathbf{u}' + \mathbf{u}'' = \mathbf{v}' + \mathbf{w}' + \mathbf{v}'' + \mathbf{w}'' = \mathbf{v}' + \mathbf{v}'' + \mathbf{w}' + \mathbf{w}''.$$

   Since V and W are subspaces, $\mathbf{v}' + \mathbf{v}'' \in V$ and $\mathbf{w}' + \mathbf{w}'' \in W$. This reveals that $\mathbf{u}' + \mathbf{u}''$ indeed belongs to $V + W$.
3. Let $\mathbf{u} \in V + W$ and $\lambda \in \mathbb{R}$. Then there exist $\mathbf{v} \in V$ and $\mathbf{w} \in W$ such that $\mathbf{u} = \mathbf{v} + \mathbf{w}$. Since

$$\lambda\mathbf{u} = \lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$$

   and V and W are closed under scalar multiplication, we see that $\lambda\mathbf{u} \in V$.

Having verified that $V + W$ has the three properties necessary to be a subspace of $\mathbb{R}^n$, we conclude that it is indeed a subspace.

6.

$$A = \begin{bmatrix} 0 & 0 & 2 \\ 2 & -1 & -1 \\ -2 & 1 & 3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\mathbf{x}) = A\mathbf{x}$. $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is a basis of $\mathbb{R}^3$.

(a) (12 points) Find $B = [T]_{\mathcal{B}}$.

Solution:

$$T(\mathbf{v}_1) = A\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} = 2\mathbf{v}_2 = 0\mathbf{v}_1 + 2\mathbf{v}_2 + 0\mathbf{v}_3 \implies [T(\mathbf{v}_1)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$T(\mathbf{v}_2) = A\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \mathbf{v}_2 + \mathbf{v}_3 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 1\mathbf{v}_3 \implies [T(\mathbf{v}_2)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$T(\mathbf{v}_3) = A\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = -\mathbf{v}_2 + \mathbf{v}_3 = 0\mathbf{v}_1 - 1\mathbf{v}_2 + 1\mathbf{v}_3 \implies [T(\mathbf{v}_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Using the column-by-column formula for the $\mathcal{B}$-matrix,

$$[T]_{\mathcal{B}} = [\, [T(\mathbf{v}_1)]_{\mathcal{B}} \; [T(\mathbf{v}_2)]_{\mathcal{B}} \; [T(\mathbf{v}_3)]_{\mathcal{B}} \,] = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) (6 points) Prove that $[T^2]_{\mathcal{B}} = B^2$.

Solution: The standard matrix of $T^2 = T \circ T$ is A^2.

Let $S = [\mathbf{v}_1\, \mathbf{v}_2\, \mathbf{v}_3]$. Then $[T^2]_{\mathcal{B}} = S^{-1}A^2S$ and $B = [T]_{\mathcal{B}} = S^{-1}AS$.

$$B^2 = (S^{-1}AS)(S^{-1}AS) = S^{-1}A(SS^{-1})AS = S^{-1}A^2S = [T^2]_{\mathcal{B}}$$

7. (6 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the invertible linear transformation such that

$$T\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Find the standard matrix for T^{-1}.

Solution:

Noting that the given data about T tells us the output vectors of $T^{-1}(\mathbf{e}_j)$, we can read the matrix A^{-1} from the input vectors specified:

$$A^{-1} = \begin{bmatrix}1 & 1 & 1\\-1 & 0 & 1\\0 & -1 & 0\end{bmatrix}.$$

8. (8 points) Prove that if A and B are n x n matrices and AB is invertible, then A and B are invertible.

Solution: There are many possible solutions. For example:

(Proof 1) If $C = AB$ is invertible, then C^{-1} exists. Thus $CC^{-1} = ABC^{-1} = A(BC^{-1}) = I_n$ Thus (thm: if $AB = I$, A and B are invertible) A is invertible with inverse BC^{-1}. Similarly, $C^{-1}C = C^{-1}AB = (C^{-1}A)B = I_n$, so B is invertible.

(Proof 2) If AB is invertible, then $\ker(AB) = \{\mathbf{0}\}$ (thm: invertibility $\Rightarrow$ kernel $= \{\mathbf{0}\}$). Then $\ker(B) = \{\mathbf{x} : B\mathbf{x} = \mathbf{0}\}$, so for any vector $\mathbf{x}$ in $\ker(B)$ we have $AB\mathbf{x} = A(B\mathbf{x}) = A\mathbf{0} = \mathbf{0}$, so $\mathbf{x} \in \ker(AB)$ and thus $\ker(B) \subset \ker(AB)$. But $\ker(AB) = \{\mathbf{0}\}$, so $\ker(B) = \{\mathbf{0}\}$ and B is invertible. Note that therefore B is also one-to-one (thm: invertibility $\Rightarrow$ transformation is one-to-one). Then for any $\mathbf{x} \in \ker(A)$ we have $A\mathbf{x} = \mathbf{0} = ABB^{-1}\mathbf{x} = AB(B^{-1}\mathbf{x})$. Thus $B^{-1}\mathbf{x} = \mathbf{0}$ (because the kernel of AB is only the zero vector), and because B^{-1} is one-to-one, $\mathbf{x} = \mathbf{0}$. Thus the kernel of A is also $\{\mathbf{0}\}$, and it, too, is invertible.

(Proof 3) Letting the columns of B be $\mathbf{b}_1, \ldots, \mathbf{b}_n$, $AB = [A\mathbf{b}_1 \cdots A\mathbf{b}_n]$. Thus the vectors $A\mathbf{b}_1, \ldots A\mathbf{b}_n$ are linearly independent (thm: invertibility $\Rightarrow$ column vectors are linearly independent), and so the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent (proved in hw), and B is invertible (thm: columns are independent $\Rightarrow$ matrix is invertible). Then $A = (AB)B^{-1}$ is a product of invertible matrices, and is thus invertible (thm: a product of invertible matrices is invertible).