Math 217 – Midterm 2

Winter 2014

Time: 120 mins.

1. Answer each question in the space provided.
2. Remember to show all your work.
3. No calculators, notes, or other outside assistance allowed.

Name: ______________________ Section: __________

Question	Points	Score
1	8	
2	15	
3	18	
4	12	
5	12	
6	18	
7	8	
8	9	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (2 points) An *orthogonal* transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.

   (b) (2 points) The *orthogonal complement* of a linear subspace V of $\mathbb{R}^n$.

   (c) (2 points) An *isomorphism* from V to W, where V and W are arbitrary linear spaces.

   (d) (2 points) The *trace* of an $n \times n$ matrix A.

2. State whether each statement is True or False and justify your answer.

   (a) (3 points) If V is a linear subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$, then for every $\mathbf{v} \in \mathbb{R}^n$, $\text{proj}_V(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_1 + \ldots + (\mathbf{v} \cdot \mathbf{v}_m)\mathbf{v}_m$.

   (b) (3 points) If an $n \times n$ matrix B may be obtained from another $n \times n$ matrix A by a series of elementary row operations, then $|\det(B)| = |\det(A)|$.

   (c) (3 points) The rows of an orthogonal matrix need not be orthonormal.

   (d) (3 points) The linear space $H = \{c_1 e^t + c_2 e^{-t} : c_1, c_2 \in \mathbb{R}\}$ is isomorphic to P_1, the linear space of all polynomials of degree ≤ 1.

   (e) (3 points) If an $n \times n$ matrix A has linearly independent columns, then $\det(A^T A) > 0$.

3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}; \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}; \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$\mathcal{B} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is a basis for a subspace V of $\mathbb{R}^4$.

(a) (5 points) $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 4 \end{bmatrix} \in V$. Find the $\mathcal{B}$-coordinates of $\mathbf{v}$.

(b) (5 points) $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix} \notin V$. Find $\text{proj}_V(\mathbf{w})$.

(c) (8 points) Find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ of $\mathbb{R}^4$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of V.

4. Let $V = \text{span}(\sin t, \cos t, e^t)$ and let $T : V \to V$ be the linear transformation defined by $T(f) = f'$.

   (a) (6 points) Let $W = \{f \in V : T(f) = f\}$. Find $\dim(W)$.

   (b) (6 points) Find the $\mathcal{B}$-matrix of T, where $\mathcal{B} = (\sin t, \cos t, e^t)$.

5. (12 points) Let $\langle -, - \rangle$ be an inner product on a linear space V and $k \in \mathbb{R}$. Consider the function $\langle\!\langle -, - \rangle\!\rangle : V \times V \to \mathbb{R}$ defined by

$$\langle\!\langle f, g \rangle\!\rangle = k \langle f, g \rangle.$$

Prove that $\langle\!\langle -, - \rangle\!\rangle$ is an inner product on V if and only if $k > 0$.

6. Let A be a 4×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x} \\ | & | & | & | \end{bmatrix},$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are fixed vectors in $\mathbb{R}^4$ and $\mathbf{x}$ is any vector in $\mathbb{R}^4$.
Let $T : \mathbb{R}^4 \to \mathbb{R}$ be the linear transformation defined as $T(\mathbf{x}) = \det(A)$.
Suppose $T(\mathbf{e}_1) = 4$, $T(\mathbf{e}_2) = 1$, $T(\mathbf{e}_3) = -1$ and $T(\mathbf{e}_4) = 2$. Explain your answers for each of the following.

(a) (3 points) What is $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right)$?

(b) (6 points) If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $B = \begin{bmatrix} - & \mathbf{x}^T & - \\ - & 2\mathbf{v}_1^T & - \\ - & 4\mathbf{v}_2^T & - \\ - & 6\mathbf{v}_3^T & - \end{bmatrix}$, what is $\det(B)$?

(c) (3 points) If A is not invertible when $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ a \\ b \end{bmatrix}$, how are a and b related?

(d) (6 points) Let $S_n(\mathbf{x}) = \det(A^n)$. Is $S_8(\mathbf{x})$ ever negative? Is there a vector $\mathbf{x}$ such that $S_5(\mathbf{x}) = 0$?

7. (8 points) Show that $T(p) = p(0)$ is a linear transformation from P_1 to $\mathbb{R}$ and find a basis for its kernel. What is $\text{rank}(T)$?

8. (9 points) Let V be a finite dimensional linear space with $\dim(V) = n$, and let $S, T : V \to \mathbb{R}$ be linear transformations such that $\ker(T) \subseteq \ker(S)$. What are the possible values of $\operatorname{rank}(T)$? Prove that $S = kT$ for some $k \in \mathbb{R}$.