Math 217 – Midterm 2

Winter 2014
Solutions

Name: ______________________________ Section: ____________

Question	Points	Score
1	8	
2	15	
3	18	
4	12	
5	12	
6	18	
7	8	
8	9	
Total:	100	

1. Write complete, precise definitions for each of the following (italicized) terms.

   (a) (2 points) An *orthogonal* transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.

   Solution : An *orthogonal transformation* $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation that preserves length.

   (b) (2 points) The *orthogonal complement* of a linear subspace V of $\mathbb{R}^n$.

   Solution : The *orthogonal complement* of a linear subspace V of $\mathbb{R}^n$ is $V^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = 0, \forall \mathbf{v} \in V\}$.

   (c) (2 points) An *isomorphism* from V to W, where V and W are arbitrary linear spaces.

   Solution : An *isomorphism* from the linear space V to the linear space W is an invertible linear transformation $T : V \to W$.

   (d) (2 points) The *trace* of an $n \times n$ matrix A.

   Solution : The *trace* of an $n \times n$ matrix A is the sum of the diagonal elements of A.

2. State whether each statement is True or False and justify your answer.

(a) (3 points) If V is a linear subspace of $\mathbb{R}^n$ with basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_m\}$, then for every $\mathbf{v} \in \mathbb{R}^n$, $\text{proj}_V(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_1 + \dots + (\mathbf{v} \cdot \mathbf{v}_m)\mathbf{v}_m$.

Solution : *False.* This is true if the $\mathbf{v}_j$ are orthonormal. For example, in $\mathbb{R}^2$ consider the projection onto the subspace V spanned by $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. An orthonormal basis is $\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so if we project a vector $\mathbf{w} = \begin{bmatrix} a \\ b \end{bmatrix}$ onto V we have $\text{proj}_V \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{u} = \begin{bmatrix} a \\ 0 \end{bmatrix}$, while $(\mathbf{v} \cdot \mathbf{w})\mathbf{v} = \begin{bmatrix} 4a \\ 0 \end{bmatrix}$.

(b) (3 points) If an $n \times n$ matrix B may be obtained from another $n \times n$ matrix A by a series of elementary row operations, then $|\det(B)| = |\det(A)|$.

Solution : *False.* If we multiply a row of A by a constant k, then $|\det(B)| = |k \det(A)|$.

(c) (3 points) The rows of an orthogonal matrix need not be orthonormal.

Solution : *False.* The columns of an orthonormal matrix A are orthonormal, and $A^T = A^{-1}$ which is also orthonormal. Columns of A^T, which are rows of A, must therefore be orthonormal.

(d) (3 points) The linear space $H = \{c_1 e^t + c_2 e^{-t} : c_1, c_2 \in \mathbb{R}\}$ is isomorphic to P_1, the linear space of all polynomials of degree ≤ 1.

Solution : *True.* We have $\dim(H) = \dim(P_1) = 2$, so they are isomorphic.

(e) (3 points) If an $n \times n$ matrix A has linearly independent columns, then $\det(A^T A) > 0$.

Solution : *True.* We have $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 \geq 0$. Because the columns of A are linearly independent, we know that it is invertible and thus $\det(A) \neq 0$.

3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}; \; \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}; \; \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$\mathcal{B} = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is a basis for a subspace V of $\mathbb{R}^4$.

(a) (5 points) $\mathbf{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \\ 4 \end{bmatrix} \in V$. Find the $\mathcal{B}$-coordinates of $\mathbf{v}$.

Solution : The $\mathcal{B}$-matrix of $\mathbf{v}$ is the vector in $\mathbb{R}^3$ whose entries are the coefficients in the unique linear combination $\mathbf{v} = x\mathbf{v}_1 + y\mathbf{v}_2 + z\mathbf{v}_3$. We find these coefficients by row-reducing the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -2 \\ -1 & -1 & 1 & 4 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -2 & 4 \\ 0 & 2 & 0 & -5 \\ 0 & 0 & 2 & 1 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & -1 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

$$\mapsto \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus

$$[\mathbf{v}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -\frac{5}{2} \\ \frac{1}{2} \end{pmatrix}$$

(b) (5 points) $\mathbf{w} = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix} \notin V$. Find $\mathrm{proj}_V(\mathbf{w})$.

Solution : Since V is 3-dimensional, the formula for the projection will be is $\mathrm{proj}_V(\mathbf{w}) = (\mathbf{w}\cdot\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{w}\cdot\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{w}\cdot\mathbf{u}_3)\mathbf{u}_3$, where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an *orthonormal* basis of V. The elements in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are mutually orthogonal, but are not unit vectors. Thus we need to normalize each of these vectors in order to obtain an orthonormal basis: $\mathbf{u}_1 = \frac{1}{2}\mathbf{v}_1$, $\mathbf{u}_2 = \frac{1}{2}\mathbf{v}_2$, and $\mathbf{u}_3 = \frac{1}{2}\mathbf{v}_3$. Thus

$$\mathrm{proj}_V(\mathbf{w}) = \frac{1}{4}\Big((\mathbf{w}\cdot\mathbf{v}_1)\mathbf{v}_1 + (\mathbf{w}\cdot\mathbf{v}_2)\mathbf{v}_2 + (\mathbf{w}\cdot\mathbf{v}_3)\mathbf{v}_3\Big)$$

$$= \frac{1}{4}\left(-6\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}\right) = \frac{1}{4}\begin{bmatrix} 0 \\ -12 \\ -4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ -1 \\ 2 \end{bmatrix}$$

(c) (8 points) Find an orthonormal basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ of $\mathbb{R}^4$ such that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal basis of V.

Solution : Note first that since the vectors in the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are mutually orthogonal, their normalizations each $\mathbf{u}_1 = \frac{1}{2}\mathbf{v}_1$, $\mathbf{u}_2 = \frac{1}{2}\mathbf{v}_2$, and $\mathbf{u}_3 = \frac{1}{2}\mathbf{v}_3$ form an orthonormal basis of V. To find $\mathbf{u}_4$, it suffices to find a vector $\mathbf{v}_4$ orthogonal to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, and then normalize $\mathbf{v}_4$. Since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are so simple, it is not difficult to find $\mathbf{v}_4$ be inspection:

$$\mathbf{v}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \qquad \text{and thus} \qquad \mathbf{u}_4 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

4. Let $V = \text{span}(\sin t, \cos t, e^t)$ and let $T : V \to V$ be the linear transformation defined by $T(f) = f'$.

(a) (6 points) Let $W = \{f \in V : T(f) = f\}$. Find $\dim(W)$.

Solution : $f \in V$ implies $f = a\sin t + b\cos t + ce^t$ for some $a, b, c \in \mathbb{R}$.
Then $T(f) = a\cos t - b\sin t + ce^t$.
If $f \in W$ then $a\sin t + b\cos t + ce^t = a\cos t - b\sin t + ce^t$, that is, $(a+b)\sin t + (b-a)\cos t = 0$. Since $\sin t, \cos t$ are linearly independent functions in V, $a + b = 0$ and $b = a$. Then $a = 0$ and $b = 0$, $W = \text{span}(e^t)$ and $\dim(W) = 1$.

(b) (6 points) Find the $\mathcal{B}$-matrix of T, where $\mathcal{B} = (\sin t, \cos t, e^t)$.

Solution : $T(\sin t) = \cos t$, $T(\cos t) = -\sin t$ and $T(e^t) = e^t$.
So $[T(\sin t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $[T(\cos t)]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ and $[T(e^t)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. (12 points) Let $\langle -,- \rangle$ be an inner product on a linear space V and $k \in \mathbb{R}$. Consider the function $\langle\langle -,- \rangle\rangle : V \times V \to \mathbb{R}$ defined by

$$\langle\langle f,g \rangle\rangle = k\langle f,g \rangle.$$

Prove that $\langle\langle -,- \rangle\rangle$ is an inner product on V if and only if $k > 0$.

Solution : Since $\langle -,- \rangle$ is an inner product on V, for any $f,g,h \in V$ and $c \in \mathbb{R}$,

(i) $\langle f,g \rangle = \langle g,f \rangle$. Then $\langle\langle f,g \rangle\rangle = k\langle f,g \rangle = k\langle g,f \rangle = \langle\langle g,f \rangle\rangle$.

(ii) $\langle f+g,h \rangle = \langle f,h \rangle + \langle g,h \rangle$. Then

$$\langle\langle f+g,h \rangle\rangle = k\langle f+g,h \rangle = k(\langle f,h \rangle + \langle g,h \rangle)k\langle f,h \rangle + k\langle g,h \rangle = \langle\langle f,h \rangle\rangle + \langle\langle g,h \rangle\rangle.$$

(iii) $\langle cf,g \rangle = c\langle f,g \rangle$. Then $\langle\langle cf,g \rangle\rangle = k\langle cf,g \rangle = k(c\langle f,g \rangle) = c(k\langle f,g \rangle) = c\langle\langle f,g \rangle\rangle$.

(iv) $\langle f,f \rangle > 0$ if $f \neq 0$. Now $\langle\langle f,f \rangle\rangle = k\langle f,f \rangle > 0$ if and only if $k > 0$.

We have shown $\langle\langle -,- \rangle\rangle$ always satisfies the symmetry and linearity conditions while it satisfies positive-definiteness if and only if $k > 0$. Thus $\langle\langle -,- \rangle\rangle$ is an inner product if and only if $k > 0$.

6. Let A be a 4×4 matrix

$$A = \begin{bmatrix} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x} \\ | & | & | & | \end{bmatrix},$$

where $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are fixed vectors in $\mathbb{R}^4$ and $\mathbf{x}$ is any vector in $\mathbb{R}^4$.
Let $T : \mathbb{R}^4 \to \mathbb{R}$ be the linear transformation defined as $T(\mathbf{x}) = \det(A)$.
Suppose $T(\mathbf{e}_1) = 4$, $T(\mathbf{e}_2) = 1$, $T(\mathbf{e}_3) = -1$ and $T(\mathbf{e}_4) = 2$. Explain your answers for each of the following.

(a) (3 points) What is $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}\right)$?

Solution : T is linear in its argument, so

$$T(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}) = T(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}) + 2T(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}) + 3T(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}) + 4T(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}) = 4 + 2(1) + 3(-1) + 4(2) = 11.$$

(b) (6 points) If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $B = \begin{bmatrix} - & \mathbf{x}^T & - \\ - & 2\mathbf{v}_1^T & - \\ - & 4\mathbf{v}_2^T & - \\ - & 6\mathbf{v}_3^T & - \end{bmatrix}$, what is $\det(B)$?

Solution : Note that three row swaps take B to $B' = \begin{bmatrix} - & 2\mathbf{v}_1^T & - \\ - & 4\mathbf{v}_2^T & - \\ - & 6\mathbf{v}_3^T & - \\ - & \mathbf{x}^T & - \end{bmatrix}$, and B'^T is the matrix A with columns multiplied by 2, 4 and 6. Thus

$$\det(B) = -\det(B') = -\det(B'^T) = -2 \cdot 4 \cdot 6 \det(A) = -48 \det(A) = -48T(\mathbf{x}).$$

Thus $\det(B) = -48 \cdot 11 = -528$.

(c) (3 points) If A is not invertible when $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ a \\ b \end{bmatrix}$, how are a and b related?

Solution : For A to not be invertible, $\det(A) = 0$. Here, $\det(A) = T(\begin{bmatrix} 1 \\ 2 \\ a \\ b \end{bmatrix}) = 1(4) + 2(1) + a(-1) + b(2) = 6 - a + 2b$. Thus we require $a = 6 + 2b$.

(d) (6 points) Let $S_n(\mathbf{x}) = \det(A^n)$. Is $S_8(\mathbf{x})$ ever negative? Is there a vector $\mathbf{x}$ such that $S_5(\mathbf{x}) = 0$?

Solution : We have $S_n(\mathbf{x}) = \det(A^n) = (\det(A))^n$. Thus $S_8(\mathbf{x}) = (\det(A))^8 \geq 0$, and $S_8(\mathbf{x})$ is never negative. Then $S_5(\mathbf{x}) = (\det(A))^5 = 0$ if $\det(A) = 0$. From (c) we know this is possible; e.g., if $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ 0 \end{bmatrix}$.

7. (8 points) Show that $T(p) = p(0)$ is a linear transformation from P_1 to $\mathbb{R}$ and find a basis for its kernel. What is $\operatorname{rank}(T)$?

Solution: T is a linear transformation because

$$\begin{aligned} T(a_1t + b_1 + a_2t + b_2) &= b_1 + b_2 = T(a_1t + b_1) + T(a_2t + b_2) \\ T(k(at + b)) &= kb = kT(b). \end{aligned}$$

The kernel of T is

$$\{at + b | T(at + b) = 0\} = \{at | a \in \mathbb{R}\}$$

and $\{t\}$ is a basis for it. By rank-nullity theorem, $rank(T) = \dim P_1 - \dim \ker T = 1$.

8. (9 points) Let V be a finite dimensional linear space with $\dim(V) = n$, and let $S, T : V \to \mathbb{R}$ be linear transformations such that $\ker(T) \subseteq \ker(S)$. What are the possible values of $\operatorname{rank}(T)$? Prove that $S = kT$ for some $k \in \mathbb{R}$.

   Solution: Since $\operatorname{im}(T) \subset \mathbb{R}$ either $\operatorname{im}(T) = \{\mathbf{0}\}$ or $\operatorname{im}(T) = \mathbb{R}$. So $\operatorname{rank}(T) \in \{0, 1\}$. Note that both possibilities could occur: if $T = 0$ then $\operatorname{rank}(T) = 0$ and if $V = \mathbb{R}$ and $T = \mathrm{id}$ then $\operatorname{rank}(T) = 1$.

   Suppose that $T = 0$. Then $\ker(T) = V$ and hence $\ker(S) = V$ as well. In this case $S = 0$ and $S = kT$ for any $k \in \mathbb{R}$. Now let's suppose that $T \neq 0$. Then by the rank nullity theorem $\ker(T)$ is $(n-1)$ dimensional. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ be a basis for $\ker(T)$. We can extend this basis to a basis $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n$ of V. Since $\mathbf{v}_n$ is not redundant, it is not in the kernel of T. Therefore $T(\mathbf{v}_n) \neq 0$ and it makes sense to set $k = S(\mathbf{v}_n)/T(\mathbf{v}_n)$. Notice that since $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \ker(T)$, they are also in the kernel of S. Let $\mathbf{v} \in V$. We can write it as

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$$

   for some $a_1, \ldots, a_n \in \mathbb{R}$. Then

$$S(\mathbf{v}) = a_1 S(\mathbf{v}_1) + \cdots + a_{n-1}S(\mathbf{v}_{n-1}) + a_n S(\mathbf{v}_n) = a_n S(\mathbf{v}_n) = a_n k T(\mathbf{v}_n) = kT(\mathbf{v}).$$

   We conclude that since $\mathbf{v}$ was arbitrary, $S = kT$.