Math 217 - Spring 2014
Quiz 2

Name : ____________________

1. (5 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 1 & 0 & 3 \end{bmatrix}.$$

Is the transformation T invertible? If so, what is the matrix of T^{-1}?

Solution : The transformation T is invertible if and only if the matrix A is invertible. The matrix A is invertible if and only if it is row equivalent to the identity matrix. To see if this is the case, and simultaneously find the matrix A^{-1} if it exists, we start with the matrix $[A|I_3]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 4 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 & 1 \end{array}\right].$$

This augmented matrix is row equivalent to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & -\frac{3}{8} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{array}\right].$$

We see that the matrix A, and so the transformation T, is invertible. Its inverse is the matrix for the transformation T^{-1}:

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{3}{8} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{4} & \frac{1}{8} & \frac{1}{4} \end{bmatrix}.$$

2. (5 points) Let A and B be two matrices of size $n \times p$, and C a matrix of size $p \times p$. Suppose you know that

$$AC = BC,$$

and that $\text{rank}(C) = p$. Is it necessarily true that $A = B$? Justify your answer.

Solution : It is true. Since $\text{rank}(C) = p$, and C is of size $p \times p$, the matrix C must be invertible. We can therefore multiply both sides of the equation by C^{-1} on the right:

$$ACC^{-1} = BCC^{-1}.$$

Since $CC^{-1} = I_p$, and $AI_p = A$, and $BI_p = B$, we have $A = B$.

3. (10 points) Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of rotation by angle $2\pi/3$. It is given by the matrix

$$A = -\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix},$$

and let $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be the horizontal shear given by the matrix

$$B = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

(a) Write the matrix for the linear transformation $T_2 \circ T_1$.

Solution : This is the matrix BA. Matrix multiplication gives

$$BA = -\frac{1}{2}\begin{bmatrix} 1 - \frac{\sqrt{3}}{2} & \sqrt{3} + \frac{1}{2} \\ -\sqrt{3} & 1 \end{bmatrix}.$$

(b) Write the matrix for the linear transformation $T_1 \circ T_2$.

Solution : This is the matrix AB. Matrix multiplication gives

$$AB = -\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} + \frac{1}{2} \\ -\sqrt{3} & 1 - \frac{\sqrt{3}}{2} \end{bmatrix}.$$