Math 217 - Spring 2014
Quiz 3 Solutions

1. (5 points) Find a basis for the kernel of the matrix below.

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

Solution : Using Gauss-Jordan elimination, you can show that the given matrix has rref $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Using rref, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \ker(A) \iff x_1 = -2x_2 + x_4$ and $x_3 = 0 \iff \mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. These vectors span the kernel and spanning vectors obtained using rref in this way are always linearly independent, by Theorem 3.2.5. Then $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis of the kernel.

2. (5 points) Find the rank and nullity of the matrix below. What does the rank-nullity theorem say about the rank and nullity of this matrix?

$$\begin{bmatrix} 1 & 1 & 3 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

Solution : Using Gauss-Jordan elimination, you can show that the given matrix has rref $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

There are 3 pivots and 1 free variable, so rank $= 3$ and nullity $= 1$. Rank-nullity theorem says rank $+$ nullity $= 4$ (number of columns), which is clearly true here.

3. (4 points) State whether each statement is True or False. Give reasons for each answer.

   (a) The columns of a 3×4 matrix are linearly dependent.
   Solution : TRUE. The 4 columns of a 3×4 matrix are vectors in $\mathbb{R}^3$. Since $\dim(V) = 3$, at most 3 vectors can be linearly independent.

   (b) If a subspace V of $\mathbb{R}^2$ does not contain the standard basis vectors $\mathbf{e}_1, \mathbf{e}_2$, then $V = \{\mathbf{0}\}$.
   Solution : FALSE. Counter-example : the line $y = x$, that is, the span of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a non-trivial subspace of $\mathbb{R}^2$ that doesn't contain $\mathbf{e}_1$ or $\mathbf{e}_2$.

4. (6 points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis of $\mathbb{R}^3$. Show that $\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}$ also form a basis of $\mathbb{R}^3$.
   Hint : Consider a linear relation between the second set of vectors. Can it be non-trivial?

   Solution : Consider any linear relation between the second set of vectors, $c_1(\mathbf{u}+\mathbf{v}) + c_2(\mathbf{v}+\mathbf{w}) + c_3(\mathbf{u}+\mathbf{w}) = \mathbf{0}$. This can be rewritten as $(c_1 + c_3)\mathbf{u} + (c_1 + c_2)\mathbf{v} + (c_2 + c_3)\mathbf{w} = \mathbf{0}$. Since $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a basis of $\mathbb{R}^3$, they are linearly independent. So only the trivial relation holds between them and this implies

$$\begin{array}{ccccccc} c_1 & & & + & c_3 & = & 0 \\ c_1 & + & c_2 & & & = & 0 \\ & & c_2 & + & c_3 & = & 0 \end{array}$$

   You can use Guass-Jordan elimination to show that this linear system has the unique solution $c_1 = c_2 = c_3 = 0$. This implies only the trivial linear relation holds between the second set of vectors, showing they are linearly independent. Since $\dim(\mathbb{R}^3) = 3$, these 3 linearly independent vectors in $\mathbb{R}^3$ must form a basis of $\mathbb{R}^3$.