Math 217 - Spring 2014
Quiz 4

Name : ____________________

1. (10 points) Let P_2 be the space of polynomials of degree at most 2, and T be the transformation $T : P_2 \to \mathbb{R}^3$ defined by

$$T(f) = \begin{bmatrix} \int_{-1}^{1} f(t)\,dt \\ f(0) \\ f(1) \end{bmatrix}.$$

(a) Write the matrix for T in the basis $\mathcal{B} = (1, t, t^2)$.

Solution : We seek the matrix A such that $Tf = A[f]_{\mathcal{B}}$ for each $f \in P_2$. The first column of A is $A\mathbf{e}_1 = T(1)$, since $[1]_{\mathcal{B}} = \mathbf{e}_1$. Similarly the second and third columns of A are $T(t)$ and $T(t^2)$, respectively. We have

$$T(1) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \qquad T(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad T(t^2) = \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix},$$

and so

$$A = \begin{bmatrix} 2 & 0 & \frac{2}{3} \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) Describe $\ker(T)$.

Solution : Let $f(t) = at^2 + bt + c$. If $Tf = \mathbf{0}$, then $\int_{-1}^{1}(at^2 + bt + c)\,dt = 0$, $f(0) = 0$, and $f(1) = 0$. In particular, $f(0) = 0$ implies that $c = 0$, so $f(t)$ is of the form $f(t) = at^2 + bt$. The condition $f(1) = 0$ implies that $b = -a$, so f is of the form $f(t) = a(t^2 - t)$. Then $\int_{-1}^{1} f(t)\,dt = \frac{2a}{3}$, and so a must be zero. Thus the only element of the kernel of T is the zero function.

(c) Is T an isomorphism? Why or why not?

Solution : Since $\ker(T) = \{0\}$, and the dimension of P_2 is the same as the dimension of $\mathbb{R}^3$, T must be an isomorphism.

2. (4 points) State whether each statement is True or False. Give reasons for each answer.

   (a) There exists a linear transformation $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ such that $\text{im}(T)$ is isomorphic to $\ker(T)$.
   Solution : False. If $\text{im}(T)$ is isomorphic to $\ker(T)$, then they must have the same dimension, and so the sum of their dimensions must be an even number. However, the dimension of $\mathbb{R}^{3\times 3}$ is $(3 \times 3) = 9$, and so the rank-nullity theorem requires that $\dim(\text{im}(T)) + \dim(\ker(T)) = 9$.

   (b) There exists a linear transformation $T : \mathbb{R}^{3\times 2} \to \mathbb{R}^{3\times 2}$ such that $\text{im}(T)$ is isomorphic to $\ker(T)$.
   Solution : True. Let T be the transformation which zeroes out the first column of a matrix in $\mathbb{R}^{3\times 2}$. That is

$$T \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \\ 0 & a_{32} \end{bmatrix}.$$

   The image of this transformation is the set of all matrices in $\mathbb{R}^{3\times 2}$ with all zeroes in the first column, and the kernel is the set of all matrices with all zeroes in the second column. These subspaces are clearly isomorphic to eachother, as they are both isomorphic to $\mathbb{R}^3$.

3. (6 points) Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three mutually orthogonal non-zero vectors in $\mathbb{R}^3$. Prove that they form a basis for $\mathbb{R}^3$. *Note: A set of vectors is mutually orthogonal if the dot product of any two distinct vectors in the set is zero.*

   Solution : The three vectors form a basis for $\mathbb{R}^3$ if and only if they are linearly independent. Consider a linear relation

$$a\mathbf{u} + b\mathbf{v} + c\mathbf{w} = \mathbf{0}, \qquad a, b, c \in \mathbb{R}.$$

   Taking dot product of both sides of this equation with $\mathbf{u}$ yields

$$\mathbf{u} \cdot (a\mathbf{u} + b\mathbf{v} + c\mathbf{w}) = \mathbf{u} \cdot \mathbf{0},$$

   equivalently

$$a(\mathbf{u} \cdot \mathbf{u}) + b(\mathbf{u} \cdot \mathbf{v}) + c(\mathbf{u} \cdot \mathbf{w}) = 0.$$

   Since $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0$, and $\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2$, this equation becomes

$$a||\mathbf{u}||^2 = 0.$$

   Since $\mathbf{u} \neq \mathbf{0}$, $||\mathbf{u}|| \neq 0$, and so we must have $a = 0$.

   Similarly, we can take dot product of both sides of the linear relation with $\mathbf{v}$ to prove that $b = 0$, and with $\mathbf{w}$ to prove that $c = 0$. Since a, b, c all must be zero, the only relation between $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ is the trivial relation and thus they are linearly independent.