Math 217 - Spring 2014
Quiz 5 Solutions

1. (a) (12 points) Find the QR factorization of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 7 & 7 & 8 \\ 1 & 2 & 1 \\ 7 & 7 & 6 \end{bmatrix}$.

Solution : Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the first, second, third columns of A respectively.

$$\|\mathbf{v}_1\| = \sqrt{1+49+1+49} = 10 \implies \mathbf{u}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{10}\begin{bmatrix} 1 \\ 7 \\ 1 \\ 7 \end{bmatrix}$$

$$\mathbf{v}_2 \cdot \mathbf{u}_1 = \tfrac{49+2+49}{10} = 10 \implies \mathbf{v}_2^{\perp} = \mathbf{v}_2 - (\mathbf{v}_2 \cdot \mathbf{u}_1)\mathbf{u}_1 = \mathbf{v}_2 - 10\mathbf{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\|\mathbf{v}_2^{\perp}\| = \sqrt{1+0+1+0} = \sqrt{2} \implies \mathbf{u}_2 = \frac{\mathbf{v}_2^{\perp}}{\|\mathbf{v}_2^{\perp}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{v}_3 \cdot \mathbf{u}_1 = \tfrac{1+56+1+42}{10} = 10 \quad \text{and} \quad \mathbf{v}_3 \cdot \mathbf{u}_2 = \tfrac{-1+0+1+0}{\sqrt{2}} = 0$$

$$\implies \mathbf{v}_3^{\perp} = \mathbf{v}_3 - (\mathbf{v}_3 \cdot \mathbf{u}_1)\mathbf{u}_1 - (\mathbf{v}_3 \cdot \mathbf{u}_2)\mathbf{u}_2 = \mathbf{v}_3 - 10\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\|\mathbf{v}_3^{\perp}\| = \sqrt{0+1+0+1} = \sqrt{2} \implies \mathbf{u}_3 = \frac{\mathbf{v}_3^{\perp}}{\|\mathbf{v}_3^{\perp}\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$$

$$Q = [\mathbf{u}_1\, \mathbf{u}_2\, \mathbf{u}_3] = \begin{bmatrix} \frac{1}{10} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{10} & \frac{1}{\sqrt{2}} & 0 \\ \frac{7}{10} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } R = \begin{bmatrix} \|\mathbf{v}_1\| & \mathbf{v}_2 \cdot \mathbf{u}_1 & \mathbf{v}_3 \cdot \mathbf{u}_1 \\ 0 & \|\mathbf{v}_2^{\perp}\| & \mathbf{v}_3 \cdot \mathbf{u}_2 \\ 0 & 0 & \|\mathbf{v}_3^{\perp}\| \end{bmatrix} = \begin{bmatrix} 10 & 10 & 10 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

(b) (3 points) For the matrix Q found in part (a), find Q^TQ.

Solution : Since Q has orthonormal columns, $Q^TQ = I_3$.

2. (5 points) Prove that the product of two orthogonal $n \times n$ matrices is orthogonal.

Solution : Let $A, B \in \mathbb{R}^{n \times n}$ be orthogonal matrices and let $T_A, T_B : \mathbb{R}^n \to \mathbb{R}^n$ be linear transformations such that T_A has standard matrix A and T_B has standard matrix B. AB is the standard matrix of $T_A \circ T_B$ and it is enough to prove this composite transformation preserves the length of each vector and is orthogonal. Since A, B are orthogonal, T_A and T_B are orthogonal transformations and they must preserve lengths of vectors.

Therefore for all $\mathbf{x} \in \mathbb{R}^n$, $\|(T_A \circ T_B)(\mathbf{x})\| = \|T_A(T_B(\mathbf{x}))\| = \|T_B(\mathbf{x})\| = \|\mathbf{x}\|$.