Math 217 - Spring 2014
Quiz 6

Name : ________________

1. (12 points) Consider the vectors $\mathbf{u} = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}1\\2\\0\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}1\\2\\3\end{bmatrix}$ in $\mathbb{R}^3$. Let A be the matrix with the column vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, i.e., $A = \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\1 & 0 & 3\end{bmatrix}$. Also, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation defined by the rule $T\mathbf{x} = \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}$, where

$$y_1 := \det \begin{bmatrix} | & | & | \\ \mathbf{x} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{bmatrix}, \quad y_2 := \det \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{x} & \mathbf{w} \\ | & | & | \end{bmatrix}, \quad y_3 := \det \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{x} \\ | & | & | \end{bmatrix}.$$

(a) Compute $\det A$. Using row reduction, we have

$$\det \begin{bmatrix}1 & 1 & 1\\1 & 2 & 2\\1 & 0 & 3\end{bmatrix} = \det \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & -1 & 2\end{bmatrix} = \det \begin{bmatrix}1 & 1 & 1\\0 & 1 & 1\\0 & 0 & 3\end{bmatrix} = 3.$$

(b) Prove that T is a linear transformation. Let

$$y_1(\mathbf{x}) := \det \begin{bmatrix} | & | & | \\ \mathbf{x} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{bmatrix}, \quad y_2(\mathbf{x}) := \det \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{x} & \mathbf{w} \\ | & | & | \end{bmatrix}, \quad y_3(\mathbf{x}) := \det \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{x} \\ | & | & | \end{bmatrix}.$$

Since the determinant is linear in each column, we have for any vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^3$, and any scalar k, $y_1(\mathbf{x}_1 + k\mathbf{x}_2) = y_1(\mathbf{x}_1) + ky_1(\mathbf{x}_2), y_2(\mathbf{x}_1 + k\mathbf{x}_2) = y_2(\mathbf{x}_1) + ky_2(\mathbf{x}_2), y_3(\mathbf{x}_1 + k\mathbf{x}_2) = y_3(\mathbf{x}_1) + ky_3(\mathbf{x}_2)$. Thus

$$T(\mathbf{x}_1 + k\mathbf{x}_2) = \begin{bmatrix}y_1(\mathbf{x}_1 + k\mathbf{x}_2)\\y_2(\mathbf{x}_1 + k\mathbf{x}_2)\\y_3(\mathbf{x}_1 + k\mathbf{x}_2)\end{bmatrix} = \begin{bmatrix}y_1(\mathbf{x}_1) + ky_1(\mathbf{x}_2)\\y_2(\mathbf{x}_1) + ky_2(\mathbf{x}_2)\\y_3(\mathbf{x}_1) + ky_3(\mathbf{x}_2)\end{bmatrix} = \begin{bmatrix}y_1(\mathbf{x}_1)\\y_2(\mathbf{x}_1)\\y_3(\mathbf{x}_1)\end{bmatrix} + k\begin{bmatrix}y_1(\mathbf{x}_2)\\y_2(\mathbf{x}_2)\\y_3(\mathbf{x}_2)\end{bmatrix} = T\mathbf{x}_1 + kT\mathbf{x}_2.$$

(c) Find the matrix for the transformation T in the standard basis, and compute its determinant. Notice that $T\mathbf{u} = \begin{bmatrix}\det A\\0\\0\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix}$. Similarly $T\mathbf{v} = \begin{bmatrix}0\\3\\0\end{bmatrix}$, and $T\mathbf{w} = \begin{bmatrix}0\\0\\3\end{bmatrix}$. If we denote the matrix for the transformation T as B, the above observations imply $BA = 3I_3$. Since A is invertible, this implies $B = 3A^{-1}$. The inverse of A is computed to be $A^{-1} = \frac{1}{3}\begin{bmatrix}6 & -3 & 0\\-1 & 2 & -1\\-2 & 1 & 1\end{bmatrix}$, and so we find $B = \begin{bmatrix}6 & -3 & 0\\-1 & 2 & -1\\-2 & 1 & 1\end{bmatrix}$. Its determinant is $\det B = \det(3A^{-1}) = 3^3 \det(A^{-1}) = 27/3 = 9$.

2. (8 points) State whether each statement is True or False. Give reasons for each answer.

(a) There exists an invertible matrix A of size 3×3 such that $\det(3A) = 3\det(A)$.
False. In general, $\det(3A) = 3^3 \det(A)$, which is only equal to $3 \det A$ if $\det A = 0$, meaning A is not invertible.

(b) For a square matrix A, $\det A = 0$ if and only if $\ker A \neq \{\mathbf{0}\}$.
True. Both $\det A = 0$ and $\ker A \neq \{\mathbf{0}\}$ are equivalent to A being noninvertible.

(c) For any invertible matrix A, $\det(A) = \det(\operatorname{rref}(A))$.
False. As a counterexample, let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ which has determinant 2. Its reduced row echelon form is the identity matrix, which has determinant 1.

(d) For any non-invertible matrix A, $\det(A) = \det(\operatorname{rref}(A))$.
True. If $A \in \mathbb{R}^{n \times n}$ is noninvertible then $\det A = 0$, and $\operatorname{rank}(A) < n$. Since $\operatorname{rref}(A)$ has the same rank as A, it is also not invertible and so has determinant zero as well.