MATH 217: LINEAR ALGEBRA SPRING 2104
WRITTEN HOMEWORK #1

Due Friday, May 9 at the beginning of class

(1) **Logical Implications:** Every mathematical statement is either true or false. Starting from given mathematical statements, we can use logical operations to form new mathematical statements which are again either true or false. Let P and Q be two statements. Here are the four most basic logical constructions:
- The statement "P *and* Q" is true if both P and Q are true statements.
- The statement "P *or* Q" is true if at least one of P or Q is true.
- The statement "*If* P, *then* Q" is true if both P and Q are true, or if P is false. The notation for "If P, then Q" is $P \Longrightarrow Q$.
- The statement "P *if and only if* Q" is true whenever both $P \Longrightarrow Q$ and $Q \Longrightarrow P$ are true statements. The notation for "P *if and only if* Q" is $P \iff Q$.

Decide whether the following statements are true or false. Justify your answers.

(a) If $9 > 5$, then pigs don't fly.
(b) If $x > 0$ and $x^2 < 0$, then $x \leq -1$.
(c) If $x > 0$, then $x^2 < 0$ or $x^3 > 0$.
(d) $x > 0$ if and only if $2x > 0$.
(e) If $1 = 2$, then Stephen Colbert is the greatest Winter Olympic athlete in history.
(f) Chickens have feathers if and only if 2 is an integer.

(2) **Negation:** The *negation* of a statement P is a statement that is true whenever P is false and false whenever P is true. The negation of P is denoted "*not* P." For example, the negation of the statement "If it is raining, then it is cloudy" is the statement "It is raining, and it is not cloudy." These statements can never be true simultaneously.

Formulate the negation of each of the statements below.

(a) The set S contains at least two integers.
(b) I eat chicken, and I don't eat beef.
(c) I love dogs, or I hate cats.
(d) If you study hard, then you will do well in this class.
(e) There is a student in class who will fail.
(f) For every girl, there is a boy who loves her.
(g) There is a real number which is larger than every rational number.

(3) **Converse and Contrapositive:** There are two additional logical statements that can be formed from a given "if-then" statement:
- The *converse* of the statement $P \Longrightarrow Q$ is the statement $Q \Longrightarrow P$. The converse may be true or false, independent of the truth value of the original "if-then" statement. (Why?)

- The *contrapositive* of the statement $P \Longrightarrow Q$ is the statement "*not* $Q \Longrightarrow$ *not* P". The original "if-then" statement and its contrapositive have the *same* truth value. (Why?)

Write both the converse and the contrapositive of the four "if - then" statements from problem 1; *i.e.* this problem has four parts.

(4) **Sets, Symbols, and Quantifiers:** We will not rigorously define what a set is, so for our purposes a *set* is a collection of objects (not necessarily numbers), which we call elements of the set. Here is some basic notation that is commonly used throughout mathematics.
- $\mathbb{R}$ is the set of real numbers, $\mathbb{Q}$ is the set of rational numbers, $\mathbb{Z}$ is the set of integers, and $\mathbb{C}$ is the set of complex numbers (which we will properly define later).
- When a set is defined in terms of its elements, we use the notation $\{\}$. For example, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- If an element a is a member of the set S, we denote this by $a \in S$.
- The term "such that" is used to specify a property that the aforementioned objects satisfy and is frequently abbreviated as "*s.t.*". In set notation, "such that" is often denoted by a vertical line $|$. For example, $\mathbb{Q} = \{\frac{a}{b} \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}$, which is read as "$\mathbb{Q}$ is the set of all fractions $\frac{a}{b}$ such that a and b are integers and b is not equal to zero."

Starting from a statement which involves a variable, we can form a new statement by quantifying the given variable.
- The quantifier "for all" indicates that something is true about every element in a given set and is denoted $\forall$. For example, the truth value of the statement $x^2 > 0$ depends on the value of x. So the quantified statement "$\forall\, x \in \mathbb{R},\ x^2 > 0$" is false, since it fails for $x = 0$.
- The quantifier "there exists" indicates that something is true for at least one element in a given set and is denoted $\exists$. For example, the truth value of the statement $x^2 = 0$ depends on the value of x. So the quantified statement "$\exists\, x \in \mathbb{R}$ s.t. $x^2 = 0$" is true, since it holds for $x = 0$.

Decide whether the following statements are true or false. No justification necessary.

(a) $\forall\, x \in \mathbb{R}$, $\forall\, y \in \mathbb{R}$, we have $x^2 + y^2 \geq 2xy$.
(b) $\forall\, x \in \mathbb{R}$, $\exists\, y \in \mathbb{Z}$ such that $y > x$.
(c) $\exists\, x \in \mathbb{R}$ s.t. $\forall\, y \in \mathbb{Z}$, $y > x$.
(d) $x \in \{\frac{a}{b} \mid a, b \in \mathbb{Z}\} \Longrightarrow x \in \mathbb{Q}$.

(5) **Subsets**: Let X and S be sets. We say that S is a subset of X, and we write $S \subseteq X$, if for all $a \in S$, we also have $a \in X$. This means that S is a set consisting of some or all of the elements in X. As we have seen, a subset may be defined according to a rule as follows

$$S = \{x \in X \mid x \text{ satisfies property } P\}.$$

The set containing no elements is called the *empty set*, denoted $\emptyset$.

(a) Use set-theoretic notation to define the half-open interval $(a, b]$ in the real numbers.
(b) Find a common English description for the following set:

$$\{a \in \mathbb{Z} \mid a = 2k + 1 \text{ for some } k \in \mathbb{Z}\}.$$

(c) Let $X = \{1, 2, 3, 4, 5\}$. How many subsets does X have?
(d) Is it true that all elements of the empty set are whistling, flying purple cows?
(e) Which of the following statements are true? (Justify your responses.)
  (i) $\emptyset \in \emptyset$.
  (ii) $\emptyset \in \{\emptyset\}$.
  (iii) $\emptyset \subseteq \{\emptyset\}$.
  (iv) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$.

(6) **Unions and Intersections**: Starting from given set, we can use set operations to form new sets.
  - Given sets X and Y, the *intersection* of X and Y is defined as
$$X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}.$$
  - Given sets X and Y, the *union* of X and Y is defined as
$$X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}.$$

(a) Let $X = \{x \in \mathbb{R} \mid -1 < x < 6\}$ and $Y = \{y \in \mathbb{R} \mid y = 2k \text{ for some } k \in \mathbb{Z}\}$. Explicitly compute $X \cap Y$ and $X \cup Y$.

(b) Let A be the xy-plane in $\mathbb{R}^3$ and B be the yz-plane in $\mathbb{R}^3$. Explicitly describe $A \cap B$ and $A \cup B$.

(*) **Bonus:** Let X, Y, Z be sets. Prove that $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$. (*Hint*: You prove that two sets A and B are equal by showing that both $A \subseteq B$ and $B \subseteq A$.)