MATH 217 SPRING 2014
WRITTEN HOMEWORK 11
SOLUTIONS

Section 5.5

Problem 10. Let $g(t) = at^2 + bt + c \in P_2$ be orthogonal to $f(t) = t$. So $\langle f, g\rangle = 0$ that is $\int_{-1}^{1} t(at^2 + bt + c)\, dt = 2b/3 = 0$ or equivalently $b = 0$. Thus $f_1 = 1 and f_2 = t^2$ form a basis of V, the space of all functions in P_2 orthogonal to $f(t) = t$. Now we apply the Gram-Schmidt algorithm to find an orthonormal basis (u_1, u_2).

$\langle f_1, f_1\rangle = \frac{1}{2}\int_{-1}^{1} 1\, dt = 1$. Then $\|f_1\| = 1$ and $u_1 = f_1 = 1$.

$\langle f_1, f_2\rangle = \frac{1}{2}\int_{-1}^{1} t^2\, dt = \frac{1}{3}$. Then $f_2^{\perp} = f_2 - \langle f_1, f_2\rangle f_1 = t^2 - \frac{1}{3}$.

$\langle f_2^{\perp}, f_2^{\perp}\rangle = \frac{1}{2}\int_{-1}^{1} (t^4 - \frac{2}{3}t^2 + \frac{1}{9})\, dt = \frac{1}{2}(\frac{2}{5} - \frac{4}{9} + \frac{2}{9}) = \frac{4}{45}$.

So $\|f_2^{\perp}\| = \frac{2}{\sqrt{45}}$ and $u_2 = \frac{\sqrt{45}}{2}(t^2 - \frac{1}{3})$.

Problem 14.

(a) For any $f, g \in P_2$ and $x \in \mathbb{R}$, $f(x)g(x) = g(x)f(x)$, so the symmetry axiom follows. For any $f, g, h \in P_2$ and $a, b, x \in \mathbb{R}$, $(af+bh)(x)g(x) = (af(x)+bh(x))g(x) = af(x)g(x)+bh(x)g(x)$, so the linearity axioms hold for $\langle -, -\rangle$.

For positive definiteness, consider $f \in P_2$ such that $\langle f, f\rangle = (f(1))^2 + (f(2))^2 = 0$. Non-negative numbers add up to 0 if and only if $f(1) = 0$ and $f(2) = 0$. Consider $f_k = k(x-1)(x-2)$ where $k \in \mathbb{R}$. So there are infinitely many polynomials $f \in P_2$ such that $f \neq 0$ but $\langle f, f\rangle = 0$ and the positive definiteness axiom fails. This is not an inner product.

(b) Symmetry and linearity axioms can be proved for $\langle\langle -, -\rangle\rangle$ just as in part (a). Similar to part (a), consider $f \in P_2$ such that $\langle\langle f, f\rangle\rangle = (f(1))^2 + (f(2))^2 + (f(3))^2 = 0$. Now $f(1) = f(2) = f(3) = 0$ and $f \in P_2$ is a polynomial of degree at most 2 with at least 3 distinct roots. The Fundamental Theorem of Algebra tells us $f = 0$. Thus we have proved $\langle\langle f, f\rangle\rangle = 0 \implies f = 0$.

Equivalently $f \neq 0 \implies \langle\langle f, f\rangle\rangle = (f(1))^2 + (f(2))^2 + (f(3))^2 > 0$ and positive definiteness holds. This is an inner product.

Problem 23. $f_1 = 1$ and $f_2 = t$ form a basis of P_1.
$\langle f_1, f_1\rangle = \frac{1}{2}(1+1) = 1$. Then $\|f_1\| = 1$ and $u_1 = f_1 = 1$.
$\langle f_1, f_2\rangle = \frac{1}{2}(0+1) = \frac{1}{2}$. Then $f_2^\perp = f_2 - \langle f_1, f_2\rangle f_1 = t - \frac{1}{2}$.
$\langle f_2^\perp, f_2^\perp\rangle = \frac{1}{2}(-\frac{1}{2}(-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2})) = \frac{1}{4}$.
So $\|f_2^\perp\| = \frac{1}{2}$ and $u_2 = 2t - 1$. Now u_1 and u_2 form an orthonormal basis of P_1.

Problem 24.

(a) $\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle = 0 + 8 = 8$.
(b) $\|g+h\|^2 = \langle g+h, g+h\rangle = \langle g, g\rangle + 2\langle g, h\rangle + \langle h, h\rangle = 1 + 2(3) + 50 = 57$. Therefore $\|g+h\| = \sqrt{57}$.
(c) Note that $\langle f, g\rangle = 0$, so f and g form an orthogonal basis for E.

$$\text{proj}_E(h) = \frac{\langle f, h\rangle}{\langle f, f\rangle} f + \frac{\langle g, h\rangle}{\langle g, g\rangle} g = 2f + 3g$$

(d) $\|f\| = \sqrt{\langle f, f\rangle} = 2$. So $u_1 = \frac{1}{2}f$. Since $\langle f, g\rangle = 0$, $g^\perp = g$. As $\langle g, g\rangle = 1$, $u_2 = g$. $h^\perp = h - \text{proj}_E(h) = h - 2f - 3g$.

$$\langle h^\perp, h^\perp\rangle = \langle h, h\rangle + 4\langle f, f\rangle + 9\langle g, g\rangle - 4\langle f, h\rangle - 6\langle g, h\rangle + 12\langle f, g\rangle = 50 + 16 + 9 - 32 - 18 = 25$$

Therefore $u_3 = \frac{1}{5}(h - 2f - 3g)$.

Section 6.1

Problem 20.

$$\begin{aligned} \det(A) &= \det\begin{bmatrix} 1 & k & 1 \\ 0 & 1 & k+1 \\ 0 & 2 & 2k+3 \end{bmatrix} && \text{operations } R_2 - R_1,\ R_3 - R_1 \\ &= \det\begin{bmatrix} 1 & k+1 \\ 2 & 2k+3 \end{bmatrix} && \text{expanding along } R_1 \\ &= 1 \end{aligned}$$

Therefore $\det(A) \neq 0$ and A is invertible for all values of k.

Problem 34. Using Theorem 6.1.5, $\det(A) = \det\begin{bmatrix} 4 & 5 \\ 3 & 6 \end{bmatrix} \det\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = 9(-5) = -45$.

Problem 56.

(a) If we swap both rows of M_2, we get I_2. Therefore $\det(M_2) = -\det(I) = -1$.
If we swap the first and last rows of M_3, we get I_3. Therefore $\det(M_3) = -\det(I) = -1$.
If we swap the first and last, second and second last rows of M_4, we get I_4. Therefore $\det(M_4) = \det(I) = 1$.
If we swap the first and last, second and second last rows of M_5, we get I_5. Therefore $\det(M_5) = \det(I) = 1$.
If we swap the first and last, second and second last, third and third last rows of M_6, we get I_6. Therefore $\det(M_6) = -\det(I) = -1$.
If we swap the first and last, second and second last, third and third last rows of M_7,

we get I_7. Therefore $\det(M_7) = -\det(I) = -1$.

(b) If n is even, we can swap R_i and R_{n+1-i} for $i = 1, \dots, n/2$ in M_n and get I_n. These are $n/2$ row swaps. Therefore $\det(M_n) = (-1)^{n/2}$.

If n is odd, we can swap R_i and R_{n+1-i} for $i = 1, \dots, (n-1)/2$ in M_n and get I_n. These are $(n-1)/2$ row swaps. (Note that the middle row remains unchanged.) Therefore $\det(M_n) = (-1)^{(n-1)/2}$.

Section 6.2

Problem 10. Use Gauss-Jordan elimination to show that the given matrix A is row-equivalent to I_5. During this process, no row swaps are needed and all pivots equal 1. The only type of elementary row operation performed is adding a multiple of one row to another row which does not change the determinant. Therefore, $\det(A) = \det(I) = 1$.

Problem 26. Let M be any 2×2 symmetric matrix in V. Then $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. So $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in V$ span V. Also $a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \implies \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \implies a = b = c = 0$.

Therefore, M_1, M_2, M_3 are linearly independent. We can now use the basis $\mathcal{B} = (M_1, M_2, M_3)$ of V to find the $\mathcal{B}$-matrix of T.

$$T(M_1) = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = 2M_1 + 2M_2 \implies [T(M_1)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

$$T(M_2) = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = 4M_1 + 4M_2 + 4M_3 \implies [T(M_2)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$$T(M_3) = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 6 \end{bmatrix} = 2M_2 + 6M_3 \implies [T(M_3)]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

Using the column-by-column formula, $B = [T]_{\mathcal{B}} = \begin{bmatrix} 2 & 4 & 0 \\ 2 & 4 & 2 \\ 0 & 4 & 6 \end{bmatrix}$.

Then $\det(T) = \det(B) = 2(24 - 8) - 4(12 - 0) = -16$.

Section 6.3

Problem 14. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 10 \\ 1 & 10 & 30 \end{bmatrix}$.

$$\det(A^T A) = (120 - 100) - (30 - 10) + (10 - 4) = 6$$

The 3-volume of the 3-parallelepiped $= \sqrt{\det(A^T A)} = \sqrt{6}$.