MATH 217 SPRING 2014
WRITTEN HOMEWORK 12
SOLUTIONS

Section 6.2

Problem 55. Suppose $A, B \in \mathbb{R}^{n\times n}$ such that B is obtained after performing an elementary row operation on A. We will prove that if $\det(B) = D(B)$, then $\det(A) = D(A)$.

Assume $\det(B) = D(B)$ and consider each of the three types of elementary row operations that could be performed on A.

(a) **Swap Rows i and j.** Then $\det(B) = -\det(A)$. But we also know D is alternating on rows, that is, $D(B) = -D(A)$. Therefore $\det(A) = D(A)$.
(b) **Divide Row i by a non-zero scalar k.** Then $\det(B) = \frac{1}{k}\det(A)$. Since D is also linear in each row, $D(B) = \frac{1}{k}D(A)$ and $\det(A) = D(A)$.
(c) **Add k times Row j to Row i, $i \neq j$.** Then $\det(B) = \det(A)$. Since D is linear in Row i of A, $D(A) = D(B) + kD(C)$ where C is obtained from A by replacing Row i by Row j. Now C has two equal rows. Swapping these rows leaves unchanged. Since D is alternating, $D(C) = -D(C)$. This implies $D(C) = 0$ and $D(A) = D(B)$. Therefore $\det(A) = D(A)$.

As a result, we see that if A such that $\det(\text{rref}(A)) = D(\text{rref}(A))$, then $D(A) = D(\text{rref}(A))$. It is enough to show that for any $A \in \mathbb{R}^{n\times n}$, $\det(\text{rref}(A)) = D(\text{rref}(A))$.

Any $A \in \mathbb{R}^{n\times n}$ is either non-invertible or invertible.

Case 1 : If A is not invertible, then $\text{rref}(A)$ has a zero row and $\det(\text{rref}(A)) = 0$. Since D is linear in the zero row which equals 0 times itself,

$$D(\text{rref}(A)) = 0D(\text{rref}(A)) = 0 = \det(\text{rref}(A)).$$

Case 2 : If A is invertible, then $\text{rref}(A) = I_n$ and $\det(\text{rref}(A)) = 1$. It is given that $D(\text{rref}(A)) = 1$.

Section 6.3

Problem 10. By the Pythagorean theorem, $\|\mathbf{v}_i^{\perp}\| \leq \|\mathbf{v}_i\|$.

We know from Theorem 6.3.3 that $|\det(A)| = \|\mathbf{v}_1\|\|\mathbf{v}_2^{\perp}\|\ldots\|\mathbf{v}_n^{\perp}\| \leq \|\mathbf{v}_1\|\|\mathbf{v}_2\|\ldots\|\mathbf{v}_n\|$.

Equality holds if and only if each $\|\mathbf{v}_i^{\perp}\| = \|\mathbf{v}_i\|$ if and only if each $\mathbf{v}_i$ is orthogonal to $\text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1})$ if and only if $\mathbf{v}_1, \ldots, \mathbf{v}_n$ form an orthogonal set.

Problem 20. T preserves orientation if and only if for all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$,

$$\det[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3] > 0 \implies \det[T(\mathbf{v}_1)\,T(\mathbf{v}_2)\,T(\mathbf{v}_3)] > 0.$$

Let A be the standard matrix of T. Note that

$$[T(\mathbf{v}_1)\,T(\mathbf{v}_2)\,T(\mathbf{v}_3)] = [A\mathbf{v}_1\,A\mathbf{v}_2\,A\mathbf{v}_3] = A[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3].$$

Then

$$\det[T(\mathbf{v}_1)\,T(\mathbf{v}_2)\,T(\mathbf{v}_3)] = \det(A)\det[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3] = \det(T)\det[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3].$$

If $\det[\mathbf{v}_1\,\mathbf{v}_2\,\mathbf{v}_3] > 0$, then $\det[T(\mathbf{v}_1)\,T(\mathbf{v}_2)\,T(\mathbf{v}_3)] > 0$ if and only if $\det(T) > 0$. Thus, T preserves orientation if and only if $\det(T) > 0$.

Section 7.1

Problem 34. We know $A\mathbf{v} = 4\mathbf{v}$.

$$(A^2 + 2A + 3I)\mathbf{v} = A(A\mathbf{v}) + 2(A\mathbf{v}) + 3\mathbf{v} = A(4\mathbf{v}) + 8\mathbf{v} + 3\mathbf{v} = 16\mathbf{v} + 11\mathbf{v} = 27\mathbf{v}$$

This shows $\mathbf{v}$ is an eigenvector of $A^2 + 2A + 3I$ and the associated eigenvalue is 27.

Problem 44. For $i = 1, \ldots, m$, there exist $\lambda_i \in \mathbb{R}$ such that $A\mathbf{e}_i = \lambda_i\mathbf{e}_i$, that is, the i^{th} column of A is $\lambda_i\mathbf{e}_i$. There are no restrictions on any entries in the last $n - m$ columns. Thus $V = \{A \in \mathbb{R}^{n\times n} : a_{ij} = 0, i \neq j, j = 1, \ldots, m\}$. Then $\dim(V) = n(n-m) + m$.

Problem 48. Let $\mathbf{v} \neq \mathbf{0}$ such that $\mathbf{v} \in \operatorname{im}(A)$. As $\operatorname{rank}(A) = \dim(\operatorname{im}(A)) = 1$ and $\mathbf{v} \neq \mathbf{0}$, $\operatorname{im}(A) = \operatorname{Span}(\mathbf{v})$. Now $A\mathbf{v} \in \operatorname{im}(A) = \operatorname{Span}(\mathbf{v})$. So there exists $\lambda \in \mathbb{R}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. As $\mathbf{v} \neq \mathbf{0}$, $\mathbf{v}$ must be an eigenvector of A.

Section 7.2

Problem 22.

$$f_{A^T}(\lambda) = \det(A^T - \lambda I) = \det((A^T - \lambda I)^T) = \det((A^T)^T - (\lambda I)^T) = \det(A - \lambda I) = f_A(\lambda)$$

We see that A and A^T have the same characteristic polynomial. This means A and its transpose have the same eigenvalues with the same algebraic multiplicities.

Problem 40.

$$\operatorname{tr}(AB) = \sum_{i=1}^{n}(AB)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij} = \sum_{j=1}^{n}(BA)_{jj} = \operatorname{tr}(BA)$$

Section 7.3

Problem 34.

(a) Suppose $B = S^{-1}AS$ and $\mathbf{x} \in \ker(B)$. Then $B\mathbf{x} = S^{-1}AS\mathbf{x} = \mathbf{0}$. Multiplying both sides by S, $A(S\mathbf{x}) = \mathbf{0}$ and $S\mathbf{x} \in \ker(A)$.
Similarly, if $\mathbf{y} \in \ker(A)$, then $S^{-1}\mathbf{y} \in \ker(B)$.

(b) If the domain of T is $\ker(B)$, then part(a) implies $\mathrm{im}(T) \subseteq \ker(A)$. So T is a function from $\ker(B)$ to $\ker(A)$. Note that S is not (necessarily) the standard matrix of T since T is not defined on $\mathbb{R}^n$, but on a subspace of $\mathbb{R}^n$. However T is a linear transformation since multiplication by S is linear.
If $\mathbf{y} \in \ker(A)$, then the $T(\mathbf{x}) = \mathbf{y}$ or $S\mathbf{x} = \mathbf{y}$ has the unique solution $\mathbf{x} = S^{-1}\mathbf{y} \in \ker(B)$. Thus T is an invertible linear transformation, that is, an isomorphism.

(c) Part (b) implies $\ker(B) \cong \ker(A)$. Then

$$\mathrm{nullity}(B) = \dim(\ker(B)) = \dim(\ker(A)) = \mathrm{nullity}(A).$$

Using the rank-nullity theorem,

$$\mathrm{rank}(B) = n - \mathrm{nullity}(B) = n - \mathrm{nullity}(A) = \mathrm{rank}(A).$$