WRITTEN HOMEWORK 1
SOLUTIONS

Problem 1: Logical implications. Decide whether the following statements are true or false. Justify your answers.

(a) If $9 > 5$, then pigs don't fly.

> Answer: True. The statement "$9 > 5$" is always true, and so is the statement "pigs don't fly."

(b) If $x > 0$ and $x^2 < 0$, then $x \leq -1$.

> Answer: True. The statement "$x^2 < 0$" is false, hence so is "$x > 0$ and $x^2 < 0$."

(c) If $x > 0$, then $x^2 < 0$ or $x^3 > 0$.

> Answer: True. Since "$x^2 < 0$ is false, the statement "$x^2 < 0$ or $x^3 > 0$" is true precisely when "$x > 0$" is true.

(d) $x > 0$ if and only if $2x > 0$.

> Answer: True. The statements "$x > 0 \Longrightarrow 2x > 0$" and "$2x > 0 \Longrightarrow x > 0$" are both true.

(e) If $1 = 2$, then Stephen Colbert is the greatest Winter Olympic athlete in history.

> Answer: True. The statement "$1 = 2$" is false.

(f) Chickens have feathers if and only if 2 is an integer.

> Answer: True. The statement "chickens have feathers" is true, and and "2 is an integer" is also true.

Problem 2: Negation. Formulate the negation of each of the statements below.

(a) The set S contains at least two integers.

> Answer: A negation is "The set S contains at most one integer."

(b) I eat chicken, and I don't eat beef.

> Answer: A negation is "I eat beef or I don't eat chicken."

(c) I love dogs, or I hate cats.

> Answer: A negation is "I don't love dogs, and I don't hate cats."

(d) If you study hard, then you will do well in this class.

> Answer: A negation is "You will study hard and you won't do well in this class."

(e) There is a student in this class who will fail.

Answer: A negation is "Every student in this class will pass." (Hopefully this is the true statement.)

(f) For every girl, there is a boy who loves her.

Answer: A negation is "There exists a girl who is not loved by any boy."

(g) There is a real number which is larger than every rational number.

Answer: A negation is "For every real number x, there is a rational number which is not less than x."

Problem 3: Converse and contrapositive. Write both the converse and the contrapositive of the following four "if-then" statements.

(a) If $9 > 5$, pigs don't fly.

Converse: If pigs don't fly, then $9 > 5$.
Contrapositive: If pigs fly, then $9 \leq 5$.

(b) If $x > 0$ and $x^2 < 0$, then $x \leq -1$.

Converse: If $x \leq -1$, then $x > 0$ and $x^2 < 0$.
Contrapositive: If $x > -1$, then $x \leq 0$ or $x^2 \geq 0$.

(c) If $x > 0$, then $x^2 < 0$ or $x^3 > 0$.

Converse: If $x^2 < 0$ or $x^3 > 0$, then $x > 0$.
Contrapositive: If $x^2 \geq 0$ and $x^3 \leq 0$, then $x \leq 0$.

(f) If $1 = 2$, then Stephen Colbert is the greatest Winter Olympic athlete in history.

Converse: If Stephen Colbert is the greatest Winter Olympic athlete in history, then $1 = 2$.
Contrapositive: If Stephen Colbert is not the greatest Winter Olympic athlete in history, then $1 \neq 2$.

Problem 4: Sets, Symbols, and Quantifiers. Decide whether the following statements are true or false.

(a) $\forall\, x \in \mathbb{R}, \forall\, y \in \mathbb{R}$, we have $x^2 + y^2 \geq 2xy$.

Answer: True. This is equivalent to $(x - y)^2 \geq 0$.

(b) $\forall\, x \in \mathbb{R}, \exists\, y \in \mathbb{Z}$ such that $y > x$.

Answer: True.

(c) $\exists\, x \in \mathbb{R}$ s.t. $\forall\, y \in \mathbb{Z}, y > x$.

Answer: False. There is no real number which is smaller than every integer.

(d) $x \in \{\frac{a}{b} \mid a, b \in \mathbb{Z}\} \implies x \in \mathbb{Q}$.

Answer: False. If $b = 0$ then $\frac{a}{b} \notin \mathbb{Q}$.

Problem 5: Subets.

(a) Use set-theoretic notation to define the half-open interval $(a, b]$ in the real numbers.

Answer: $\{x \in \mathbb{R} \mid a < x \leq b\}$.

(b) Find a common English description for the following set:

$$\{a \in \mathbb{Z} \mid a = 2k + 1 \text{ for some } k \in \mathbb{Z}\}.$$

Answer: This is the set of all integers which are not divisible by 2, commonly called the odd integers.

(c) Let $X = \{1, 2, 3, 4, 5\}$. How many subsets does X have?

Answer: It has $2^5 = 32$ subsets.

(d) Is it true that all elements of the empty set are whistling, flying purple cows?

Answer: It is indeed true.

(e) Which of the following statements are true? (Justify your responses.)

(i) $\emptyset \in \emptyset$.

Answer: False. The empty contains no elements.

(ii) $\emptyset \in \{\emptyset\}$.

Answer: True. The set of containing one element, the empty set, does contain the empty set.

(iii) $\emptyset \subseteq \{\emptyset\}$.

Answer: True. The empty set is a subset of every set.

(iv) $\{\emptyset\} \subseteq \{\emptyset, \{\emptyset\}\}$.

Answer: True. The first element of the set on the right is the empty set, so the set containing the empty set is a subset of the set on the right.

Problem 6: Unions and intersections.

(a) Let $X = \{x \in \mathbb{R} \mid -1 < x < 6\}$ and $Y = \{y \in \mathbb{R} \mid y = 2k \text{ for some } k \in \mathbb{Z}\}$. Explicitly compute $X \cap Y$ and $X \cup Y$.

Answer: $X \cap Y = \{0, 2, 4\}$, and
$X \cup Y = \{x \in \mathbb{R} \mid -1 < x \leq 6, \text{ or } x \text{ is an even integer.}\}$

(b) Let A be the xy-plane in $\mathbb{R}^3$ and B be the yz-plane in $\mathbb{R}^3$. Explicitly describe $A \cap B$ and $A \cup B$.

Answer: $A \cap B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ and } z = 0\}$, and
$A \cup B = \{(x, y, z) \in \mathbb{R}^3 \mid x = 0 \text{ or } z = 0\}$.

(*) **Bonus:** Let X, Y, Z be sets. Prove that $(X \cap Y) \cup Z = (X \cup Z) \cap (Y \cup Z)$. (*Hint*: You prove that two sets A and B are equal by showing that both $A \subseteq B$ and $B \subseteq A$.)

Proof: Denote $A = (X \cap Y) \cup Z$, and $B = (X \cup Z) \cap (Y \cup Z)$. Let a be an arbitrary element of A. We would like to show that it is also in B. From the definition, a is either in both X and Y, or it is in Z. Notice that since $X \subseteq X \cup Z$, $Z \subseteq X \cup Z$ and $Y \subseteq Y \cup Z$, $Z \subseteq Y \cup Z$, we have the following statements:

$$X \cap Y \subseteq (X \cup Z) \cap (Y \cup Z) = B,$$

$$Z \subseteq (X \cup Z) \cap (Y \cup Z) = B.$$

It follows that if $a \in X \cap Y$, then a is definitely in B, and also if $a \in Z$, then a is definitely in B. Since a was an arbitrary element of A, we have proved

$$A \subseteq B.$$

Now let b be an arbitrary element of B. We would like to show that it is also in A. If $b \in Z$, then b is clearly in A, since $Z \subseteq A$. If $b \notin Z$, then b must be in X, since $B \subseteq X \cup Z$, and b must also be in Y, since $B \subseteq Y \cup Z$. Thus $b \in X \cap Y \subseteq A$. This proves that

$$B \subseteq A.$$

Since we have shown that $A \subseteq B$ and $B \subseteq A$, we must have $A = B$.