WRITTEN HOMEWORK 2
SOLUTIONS

1. Section 1.1

Problem 20. For $k = 2$, the system has infinitely many solutions. Otherwise, the system of equations is equivalent to the system

$$(k+2)x = k+5$$
$$(k+2)y = k$$
$$(k+2)z = 1$$

This system has no solution (and is thus inconsistent) when $k = -2$. Otherwise it has the unique solution

$$x = \frac{k+5}{k+2}, \quad y = \frac{k}{k+2}, \quad z = \frac{1}{k+2}.$$

Problem 30. The relevant system of equations is

$$T_1 = \frac{200+0+0+T_2}{4}$$
$$T_2 = \frac{200+0+T_1+T_3}{4}$$
$$T_3 = \frac{400+0+0+T_2}{4},$$

which can be written in the form

$$\begin{aligned} T_1 - \frac{1}{4}T_2 \qquad\qquad &= \quad 50 \\ -\frac{1}{4}T_1 + T_2 \quad -\frac{1}{4}T_3 &= \quad 50 \\ -\frac{1}{4}T_2 \quad +T_3 &= \quad 100, \end{aligned}$$

and solved to find

$$T_1 = 75, \qquad T_2 = 100, \qquad T_3 = 125.$$

2. Section 1.2

Problem 19. Since a reduced row echelon matrix of size 4×1 matrix has only one column, it can have at most one leading one. If there is a leading one, it must be in the first entry, since otherwise there would be a row of zeroes above a row with a leading one. Thus the two possibilities are

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

Problem 22. If there are two leading ones, then a 2×2 rref matrix must be of the form

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}.$$

If there is only one leading one, it must be in the first row, although it may be in either column, giving the types

$$\begin{pmatrix}1 & *\\0 & 0\end{pmatrix}, \qquad \begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}.$$

There may also be no leading ones, in which case the matrix is the zero matrix:

$$\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}.$$

In all this gives 4 types of 2×2 rref matrices.

Problem 39. The relevant system of equations is

$$\begin{aligned} x_1 &= 0.2x_2 + 0.3x_3 + 320\\ x_2 &= 0.1x_1 + 0.4x_3 + 90\\ x_3 &= 0.2x_1 + 0.5x_2 + 150\,, \end{aligned}$$

which can be written in the form

$$\begin{aligned} x_1 - 0.2x_2 \quad -0.3x_3 &= \quad 320\\ -0.1x_1 + x_2 \quad\quad -0.4x_3 &= \quad 90\\ -0.2x_1 - 0.5x_2 \quad\quad +x_3 &= \quad 150\,. \end{aligned}$$

The augmented matrix is

$$\begin{pmatrix}1 & -0.2 & -0.3 & \vdots & 320\\ -0.1 & 1 & -0.4 & \vdots & 90\\ -0.2 & -0.5 & 1 & \vdots & 150\end{pmatrix},$$

which is equivalent to the rref matrix

$$\begin{pmatrix}1 & 0 & 0 & \vdots & 500\\ 0 & 1 & 0 & \vdots & 300\\ 0 & 0 & 1 & \vdots & 400\end{pmatrix},$$

from which we find $x_1 = 500$, $x_2 = 300$, and $x_3 = 400$.

3. Section 1.3

Problem 22. If the rank of the coefficient matrix is less than 3, then the system is inconsistent. Thus the rank must be 3, as it cannot be greater than 3. The rref for a 3×3 matrix with rank 3 is necessarily

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 28. The three leading ones in a 5×3 rref matrix with rank 3 must come in the first three rows, and the only way to put them there is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 47.

(a) All homogeneous systems are consistent. This is true because they all have the solution $x_1 = x_2 = \cdots = x_m = 0$.

(b) Denote by m the number of unknowns, and n the number of equations, and by A the coefficient matrix of the system, which is of size $n \times m$. If $n < m$, then $\text{rank}(A) < m$, since $\text{rank}(A) \leq m$. Thus there are fewer leading ones in $\text{rref}(A)$ than there are columns, and therefore there is at least one free variable. Since the system is consistent, this implies that there are infinitely many solutions.

(c) Assume $A\vec{x_1} = \vec{0}$, and $A\vec{x_2} = \vec{0}$. Using matrix algebra we have

$$A(\vec{x_1} + \vec{x_2}) = A\vec{x_1} + A\vec{x_2} = \vec{0} + \vec{0} = \vec{0},$$

thus $\vec{x_1} + \vec{x_2}$ is also a solution to the homogeneous system.

(d) Assume $A\vec{x_1} = \vec{0}$, and $k \in \mathbb{R}$. Again using the matrix algebra,

$$A(k\vec{x_1}) = kA\vec{x_1} = k\vec{0} = \vec{0},$$

thus $k\vec{x_1}$ is also a solution to the homogeneous system.

Problem 48.

(a) Assume $A\vec{x_1} = \vec{b}$, and $A\vec{x_h} = \vec{0}$. Using matrix algebra we have

$$A(\vec{x_1} + \vec{x_h}) = A\vec{x_1} + A\vec{x_h} = \vec{b} + \vec{0} = \vec{b},$$

thus $\vec{x_1} + \vec{x_h}$ is a solution to the non-homogeneous system.

(b) Assume $A\vec{x_1} = \vec{b}$, and $A\vec{x_2} = \vec{b}$. Using matrix algebra we have

$$A(\vec{x_1} - \vec{x_2}) = A\vec{x_1} - A\vec{x_2} = \vec{b} - \vec{b} = \vec{0},$$

thus $\vec{x_1} - \vec{x_2}$ is a solution to the homogeneous system.

(c) The line should be parallel to the line of solutions of $A\vec{x} = \vec{0}$, passing through $\vec{x_1}$.

Problem 58. This amounts to determining for which values of b and c the linear system

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ b \\ c \end{pmatrix}$$

is consistent. The augmented matrix is

$$\begin{pmatrix} 1 & 2 & -1 & \vdots & 3 \\ 3 & 6 & -3 & \vdots & b \\ 2 & 4 & -2 & \vdots & c \end{pmatrix},$$

which is row equivalent to the matrix

$$\begin{pmatrix} 1 & 2 & -1 & \vdots & 3 \\ 0 & 0 & 0 & \vdots & b-9 \\ 0 & 0 & 0 & \vdots & c-6 \end{pmatrix},$$

from which we see that the system is consistent only for $b = 9$ and $c = 6$. Thus the vector $\begin{pmatrix} 3 \\ b \\ c \end{pmatrix}$ is a linear combination of the three vectors in question only for $b = 9$ and $c = 6$.