MATH 217 SPRING 2014
WRITTEN HOMEWORK 3
SOLUTIONS

Section 1.2

Problem 26. Yes. If the matrix A was transformed into the matrix B by means of an elementary row operation, then there is an inverse elementary row operation that transforms B into A. The inverse operation is another elementary row operation of the same type. Let's consider each of the 3 types :

- Operation = Swap rows i and j. Inverse = Swap rows i and j.
- Operation = (row i)$/k$, where k is a non-zero scalar.
  Inverse = k(row i) = (row i)$/(1/k)$.
- Operation = (row i) $-$ k(row j), where $i \neq j$. Inverse = row i + k (row j).

Problem 27. Yes. Invert each elementary row operation in the sequence(see solution to 1.2.26), in the reverse order. That is, invert the last operation performed in the original sequence, then the second last and so on.

Section 1.3

Problem 44. Consider the reduced row-echelon form of A. Since this is an $n \times m$ matrix with $n > m$, all rows cannot have pivots and there must be a row of 0s. Let row i be the first row of 0s. Augment rref(A) with the standard basis vector $\mathbf{e}_i$. This matrix is in reduced row-echelon form and it is the augmented matrix of an inconsistent system.

Gauss Jordan elimination can be performed on A to obtain rref(A).This is a sequence of elementary row operations. Applying the inverse sequence (see solution to 1.2.27) to $[\text{rref}(A)|\mathbf{e}_i]$ yields a matrix of the form $[A|\mathbf{b}]$, where $\mathbf{b}$ is some vector in $\mathbb{R}^n$. Note that the reduced row-echelon form of $[A|\mathbf{b}]$ must be $[\text{rref}(A)|\mathbf{e}_i]$. Therefore, the system $A\mathbf{x} = \mathbf{b}$ must be inconsistent.

Problem 45. Let $A \in \mathbb{R}^{n\times m}$, $k \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^m$. Recall that $k \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} kx_1 \\ \vdots \\ kx_m \end{bmatrix}$.

The i^{th} entry of $A(k\mathbf{x}) = \sum_{j=1}^{m} a_{ij}(kx_j) = \sum_{j=1}^{m} k(a_{ij}x_j) = k(\sum_{j=1}^{m} a_{ij}x_j) = i^{th}$ entry of $k(A\mathbf{x})$, for each $i \in \{1, 2, \ldots, n\}$ Therefore, the two vectors are equal.

Problem 50. $[A|\mathbf{b}]$ is a 4×4 matrix with rank 4, so its reduced row-echelon form must be I_4. The last row corresponds to the equation $0 = 1$. Therefore, the system has no solutions.

Section 2.1

Problem 39. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then there exists $A \in \mathbb{R}^{n\times m}$ such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$. By Theorem 2.1.2, $A = [T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_m)]$.

For all $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$, $T(\mathbf{x}) = A\mathbf{x} = [T(\mathbf{e}_1) \ \dots \ T(\mathbf{e}_m)] \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 T(\mathbf{e}_1) + \dots + x_m T(\mathbf{e_m})$, using Theorem 1.3.8.

Problem 43.

(a) $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \mathbf{x}$. Let $A = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$, then $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$. By definition, T is linear and A is its matrix.

(b) Similarly, for all $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$, $T(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x} = \mathbf{v}^T\mathbf{x}$. Then T is linear with matrix $\mathbf{v}^T$.

(c) Suppose $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation. Then there exist $A \in \mathbb{R}^{1\times 3}$ such that for all $\mathbf{x} \in \mathbb{R}^3$, $T(\mathbf{x}) = A\mathbf{x} = A^T \cdot \mathbf{x}$. Note that $A^T \in \mathbb{R}^{3\times 1}$, that is, it is a vector in $\mathbb{R}^3$.

Section 2.2

Problem 24.

(a) If $A = [\mathbf{v}\ \mathbf{w}]$, then $\mathbf{v} = T(\mathbf{e}_1)$ and $\mathbf{w} = T(\mathbf{e}_2)$. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ both have length 1 and T preserves lengths of vectors, $\mathbf{v}$ and $\mathbf{w}$ must also be unit vectors. Since $\mathbf{e}_1$ and $\mathbf{e}_2$ are perpendicular to each other and T also preserves angles between vectors, $\mathbf{v}$ and $\mathbf{w}$ must be perpendicular to each other.

(b) Let $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} x \\ y \end{bmatrix}$. From part (a), we know that $a^2 + b^2 = 1$, $x^2 + y^2 = 1$ and $\mathbf{v} \cdot \mathbf{w} = ax + by = 0$. Given the restriction $x^2 + y^2 = 1$, there are only two possible solutions to $ax + by = 0$, namely, $x = -b, y = a$ or $x = b, y = -a$. So $\mathbf{w}$ is either $\begin{bmatrix} -b \\ a \end{bmatrix}$ or $\begin{bmatrix} b \\ -a \end{bmatrix}$.

(c) Part (b) implies A is either $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ or $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. The first matrix represents a rotation while the second represents a reflection about a line.

Section 2.3

Problem 27. Let A, B be $n \times m$ matrices and let C, D be $m \times p$ matrices. For any $i \in \{1, \ldots n\}$ and $j \in \{1, \ldots, p\}$,

$$\begin{aligned}
(A(C+D))_{ij} &= \sum_{k=1}^{m} a_{ik}(C+D)_{kj} \\
&= \sum_{k=1}^{m} a_{ik}(c_{kj} + d_{kj}) \\
&= \sum_{k=1}^{m} a_{ik}c_{kj} + \sum_{k=1}^{m} a_{ik}(d_{kj} \\
&= (AC)_{ij} + (AD)_{ij} = (AC+AD)_{ij}
\end{aligned}$$

Since all their corresponding entries are equal, the matrices $A(C+D)$ and $AC+AD$ must be equal.

Similarly, for any $i \in \{1, \ldots n\}$ and $j \in \{1, \ldots, p\}$,

$$\begin{aligned}
((A+B)C))_{ij} &= \sum_{k=1}^{m} (A+B)_{ik}c_{kj} \\
&= \sum_{k=1}^{m} (a_{ik} + b_{ik})c_{kj} \\
&= \sum_{k=1}^{m} a_{ik}c_{kj} + \sum_{k=1}^{m} b_{ik}c_{kj} \\
&= (AC)_{ij} + (BC)_{ij} = (AC+BC)_{ij}
\end{aligned}$$

Therefore $(A+B)C = AC + BC$.

Problem 84. Suppose there exists an $m \times n$ matrix $X = [\mathbf{x}_1 \ \ldots \ \mathbf{x}_n]$, where the column vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^m$, such that $AX = I_n$. Note that $I_n = [\mathbf{e}_1 \ \ldots \ \mathbf{e}_n]$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors of $\mathbb{R}^n$. By Theorem 2.3.2, $[A\mathbf{x}_1 \ \ldots \ A\mathbf{x}_n] = [\mathbf{e}_1 \ \ldots \ \mathbf{e}_n]$. Comparing columns, we can see that a solution X to $AX = I$ exists if and only if $A\mathbf{x}_i = \mathbf{e}_i$ is consistent for each $i = 1, \ldots, n$. A is an $n \times m$ matrix with rank n, so there is no row of 0s in rref(A), so any equation of the form $A\mathbf{x} = \mathbf{b}$ must be consistent. This proves that there exists at least one solution for each column of X.

If $n < m$, rank$(A) < m$ and there must be free variables. Each equation of the form $A\mathbf{x} = \mathbf{b}$ must have infinitely many solutions. There are infinitely many choices for each column of X, so there are infinitely many matrices X such that $AX = I$.