MATH 217 SPRING 2014
WRITTEN HOMEWORK 4
SOLUTIONS

Section 2.1

Problem 13.

(a) If $a \neq 0$, then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & b/a \\ 0 & d-(bc)/a \end{bmatrix}$. If $ad - bc = 0$, then there is a row of 0s in rref and the matrix is not invertible. If $ad - bc = 0$, then the matrix is invertible.

If $a = 0$ and $c = 0$, then the matrix has a column of 0s and it cannot be invertible. If $a = 0$ and $c \neq 0$, then $\begin{bmatrix} 0 & b \\ c & d \end{bmatrix}$ is row-equivalent to $\begin{bmatrix} 1 & d/c \\ 0 & b \end{bmatrix}$. In this case $ad - bc \neq 0$ if and only if $bc \neq 0$ if and only if $b \neq 0$. Observe that there is a second pivot and the matrix is invertible if and only if $b \neq 0$.

(b) Suppose $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. From (a), we know $ad-bc \neq 0$. Let $B = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. It is easy to check that $AB = I$ and $BA = I$. So B is the inverse of A.

Alternate Solution : We can show that if $a \neq 0$, $\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$ is row-equivalent to $\left[\begin{array}{cc|cc} 1 & b/a & 1/a & 0 \\ 0 & d-(bc)/a & -c/a & 1 \end{array}\right]$. There is no second pivot if $ad - bc = 0$ and the matrix is not invertible.

If $ad-bc \neq 0$, then the augmented matrix is row-equivalent to $\left[\begin{array}{cc|cc} 1 & 0 & d/(ad-bc) & -b/(ad-bc) \\ 0 & 1 & -c/(ad-bc) & a/(ad-bc) \end{array}\right]$. The inverse must be $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

If $a = 0$ and $c = 0$, there is no second pivot and the matrix is not invertible. If $a = 0$ and $c \neq 0$, $\left[\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$ is row-equivalent to $\left[\begin{array}{cc|cc} 1 & d/c & 0 & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right]$.

If $b = 0$, there is no second pivot and the matrix is not invertible. If $b \neq 0$, the augmented matrix is row-equivalent to $\left[\begin{array}{cc|cc} 1 & 0 & -d/bc & 1/c \\ 0 & 1 & 1/b & 0 \end{array}\right]$. Then the inverse is $\dfrac{1}{-bc}\begin{bmatrix} d & -b \\ -c & 0 \end{bmatrix}$ which is of the form $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ when $a = 0$ and $ad - bc = -bc \neq 0$.

Problem 58.

(a) Let the masses of platinum and silver be x and y grams respectively. Then

$$\begin{array}{ccccc} x & + & y & = & 5000 \\ \frac{x}{20} & + & \frac{y}{10} & = & 370 \end{array}$$

The augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & 5000 \\ 1/20 & 1/10 & 370 \end{array}\right]$ has rref $\left[\begin{array}{cc|c} 1 & 0 & 2600 \\ 0 & 1 & 2400 \end{array}\right]$.

The mass of platinum is 2600 grams and the mass of silver is 2400 grams, which means there is only 52% platinum and the goldsmith is a crook.

(b) From part (a), $A = \begin{bmatrix} 1 & 1 \\ 1/20 & 1/10 \end{bmatrix}$.

(c) A is a 2×2 matrix with rank 2, so A is invertible. Alternatively using 2.1.13, A is invertible because $ad - bc = 1/20 \neq 0$.

$$A^{-1} = \begin{bmatrix} 2 & -20 \\ -1 & 20 \end{bmatrix}$$

The unique solution to $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5000 \\ 370 \end{bmatrix}$ must be $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} 5000 \\ 370 \end{bmatrix} = \begin{bmatrix} 2600 \\ 2400 \end{bmatrix}$.

Section 2.2

Problem 8. $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ with $a = 0$ and $b = -1$. Since $a^2 + b^2 = 1$, this matrix represents a reflection about a line. Comparing with the form $\begin{bmatrix} 2u_1^2 - 1 & 2u_1u_2 \\ 2u_1u_2 & 2u_2^2 - 1 \end{bmatrix}$, we see that $\mathbf{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ or $\mathbf{u} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$. The line is $y = -x$.

Problem 12.

(a) Since A is a reflection matrix, $A(A\mathbf{x}) = \mathbf{x}$.

(b) $A\mathbf{v} = A(\mathbf{x} + A\mathbf{x}) = A\mathbf{x} + A(A\mathbf{x}) = A\mathbf{x} + \mathbf{x} = \mathbf{v}$.

(c) $A\mathbf{w} = A(\mathbf{x} - A\mathbf{x}) = A\mathbf{x} - A(A\mathbf{x}) = A\mathbf{x} - \mathbf{x} = -\mathbf{w}$.

(d) If $\mathbf{v}$ is a non-zero vector and its reflection is itself, then $\mathbf{v}$ must be parallel to the line of reflection. If $\mathbf{w}$ is a non-zero vector and its reflection is the opposite vector $-\mathbf{w}$, then $\mathbf{w}$ must be perpendicular to the line of reflection. So the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\pi/2$.

(e) See part (d).

Problem 20. Let T be the reflection about the xz-plane. Since $\mathbf{e}_1$ and $\mathbf{e}_3$ are parallel to the xz-plane and $\mathbf{e}_2$ is perpendicular to the xz-plane, $T(\mathbf{e}_1) = \mathbf{e}_1$, $T(\mathbf{e}_2) = -\mathbf{e}_2$ and $T(\mathbf{e}_3) = \mathbf{e}_3$. Using Theorem 2.1.2, the matrix of T is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Alternate Solution : Let T be the reflection about the xz-plane.

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

By definition, the matrix of T must be $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Problem 26.

(a) The scaling factor is 4, so $A = 4I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(b) The projected vector $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ is parallel to the x-axis, so B is the matrix of orthogonal projection onto the x-axis. $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

(c) Rotation matrix C is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $C\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} -5b \\ 5a \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

This implies $b = -0.6$, $a = 0.8$ and $C = \begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$

(d) Since the second co-ordinate remains the same after transformation, D represents a horizontal shear. D is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ and $D\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+3k \\ 3 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

This implies $k = 2$ and $D = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

(e) Reflection matrix E is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ and $E\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7a+b \\ 7b-a \end{bmatrix}$.

The linear system $\begin{bmatrix} 7 & 1 \\ -1 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$ has the solution $a = -0.8$ and $b = 0.6$ and $E = \begin{bmatrix} -0.8 & 0.6 \\ 0.6 & 0.8 \end{bmatrix}$.

Section 2.3

Problem 14. The products $AB, AC, AD, AE, BA, B^2, BE, CA, CB, CE, DA, DC, D^2, EA, EC, ED$ are not defined because the number of columns in the first matrix does not equal number of rows in the second matrix.

$$A^2 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \qquad BC = \begin{bmatrix} 14 & 8 & 2 \end{bmatrix} \qquad BD = \begin{bmatrix} 6 \end{bmatrix}$$

$$C^2 = \begin{bmatrix} -2 & -2 & -2 \\ 4 & 1 & -2 \\ 10 & 4 & -2 \end{bmatrix} \qquad CD = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \qquad DB = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \qquad DE = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}$$

$$EB = \begin{bmatrix} 5 & 10 & 15 \end{bmatrix} \qquad E^2 = \begin{bmatrix} 25 \end{bmatrix}$$

Problem 38.

$$A^2 = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \qquad A^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A^4 = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}$$

The pattern is $A^{3k} = I_2, \quad A^{3k+1} = A, \quad A^{3k+2} = A^2$ for all $k \in \mathbb{N}$.

A has the form of a rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with $\cos\theta = -1/2$ and $\sin\theta = \sqrt{3}/2$, therefore the angle of rotation $\theta = 2\pi/3$. Rotation by $4\pi/3$ has matrix $A^2 = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$. Rotation by $6\pi/3 = 2\pi$ is the identity transformation and it has matrix $A^3 = I_2$. Then the pattern repeats.

Problem 46. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an orthogonal projection onto a line L through the origin. Then $T(T(\mathbf{x})) = T(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^2$. A^2 is the matrix of $T \circ T = T^2$, but $T^2 = T$ implies $A^2 = A$. We can choose the projection matrix for any line such that all entries are non-zero, for example, $\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ which represents projection onto $y = x$.

Section 2.4

Problem 20. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that $\mathbf{y} = T(\mathbf{x})$. Then the matrix of T is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 8 \\ 2 & 7 & 12 \end{bmatrix}$$

Performing Gauss-Jordan elimination on $\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 8 & 0 & 1 & 0 \\ 2 & 7 & 12 & 0 & 0 & 1 \end{array}\right]$ we see that the augmented matrix has reduced row-echelon form $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -8 & -15 & 12 \\ 0 & 1 & 0 & 4 & 6 & -5 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{array}\right]$.

Therefore

$A^{-1} = \begin{bmatrix} -8 & -15 & 12 \\ 4 & 6 & -5 \\ -1 & -1 & 1 \end{bmatrix}$ is the matrix of the inverse linear transformation $T^{-1} : \mathbb{R}^3 \to \mathbb{R}^3$ and the inverse transformation is

$$\begin{array}{ccccccc} x_1 & = & -8y_1 & - & 15y_2 & + & 12y_3 \\ x_2 & = & 4y_1 & + & 6y_2 & - & 5y_3 \\ x_3 & = & -y_1 & - & y_2 & + & y_3 \end{array}$$

Problem 30. $\begin{bmatrix} 0 & 1 & b \\ -1 & 0 & c \\ -b & -c & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ -b & -c & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & -c & -bc \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -c \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$

These matrices are row-equivalent for all values of b and c. The rank of the matrix is always 2 and rref $\neq I_3$. So it is never invertible.

Problem 78. We know that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \end{bmatrix}$ and $A \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Using theorem 2.3.2, we can see that $AB = C$ where $B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix}$.

As $\det(B) = 1(5) - 2(2) = 1 \neq 0$, B is invertible and $B^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$.

$$AB = C \quad \Leftrightarrow \quad A = CB^{-1} = \begin{bmatrix} 7 & 1 \\ 5 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 33 & -13 \\ 21 & -8 \\ 9 & -3 \end{bmatrix}$$