MATH 217 SPRING 2014
WRITTEN HOMEWORK 5
SOLUTIONS

Section 2.4

Problem 35.

(a) If all of the diagonal terms a, d, and f are nonzero then the matrix A is already in row echelon form, and we can see that $\text{rank}(A) = 3$. It follows that A is inveritble by Theorem 2.4.3. Conversely if any of the diagonal terms are zero, then there are less than 3 pivots, and so $\text{rank}(A) < 3$, thus A is not inveritible, again by Theorem 2.4.3.

(b) By an argument similiar to the one in part (a), an upper triangular matrix is invertible if and only if all diagonal entries are nonzero.

(c) The inverse of an upper triangular matrix is also upper triangular. An upper triangular matrix of size $n \times n$ with nonzero entries along the diagonal can be transformed to reduced row echelon form in the following way: First multiply the last row by a constant so that the row vector in the last row is $\mathbf{e_n}$. Add a multiple of the last row to each of the other rows so that the last column is also $\mathbf{e_n}$. Then multiply the $(n-1)$st row by a constant so that it is $\mathbf{e_{n-1}}$, and add multiples of this row to rows above it so that the $(n-1)$st column is $\mathbf{e_{n-1}}$ as well. Continuing in this manner, we can reduce the jth row to the vector $\mathbf{e_j}$ for each $j = n, n-1, \ldots, 1$. Notice that the only row operations necessary are multiplication of a row by a constant, and addition of a multiple of a lower row to a higher row. Performing these row operations on the identity matrix necessarily yields an upper triangular matrix. Thus the inverse of an upper triangular matrix is upper triangular as well.

(d) A lower triangular matrix is invertible exactly when its diagonal entries are nonzero. Consider a lower triangular matrix A, and think of it as the coefficient matrix for an $n \times n$ linear system with unknowns $(x_1, x_2, \ldots, x_n)$. Then A is invertibe if and only if the equation

$$A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \mathbf{0}$$

has a unique solution. Now let A_1 be the matrix which is obtained from A by interchanging columns in the following way: the jth column of A is the $(n-j)$th column of

A_1. Then the linear equation above is equivalent to the linear system

$$A_1 \begin{bmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{bmatrix} = \mathbf{0},$$

and so A is invertible if and only if A_1 is invertible. Notice that A_1 has all zero entries above its anti-diagonal, and that the diagonal entries of A are the anti-diagonal entries of A_1. The matrix A_1 can be transformed to an upper triangular matrix A_2 by interchanging rows, and the anti-diagonal entries of A_1 are the diagonal entries of A_2. By part (b) we know that A_2 is invertible if and only if its diagonal entries are non-zero. Since A_2 is row equivalent to A_1, it follows that A_1 is invertible if and only if its anti-diagonal entries are nonzero, or equaivalently when the diagonal entries of A are nonzero.

Problem 42. Permutation matrices are invertible. Since each row and column contain exactly one 1, any permutation matrix is row equivalent to the identity matrix via interchange of rows. Likewise, starting from the identity matrix, any interchange of rows yields a permutation matrix. This implies that the inverse of a permutation matrix is also a permutation matrix.

Problem 99. Suppose A is invertible, and that $A^2 = A$. Then we can multiply both sides of the equation $A^2 = A$ on the right by A^{-1} to get $A^2A^{-1} = AA^{-1}$. Since $AA^{-1} = I_n$, the right hand side is I_n. The left hand side is $A^2A^{-1} = A(AA^{-1}) = A(I_n) = A$. Therefore we must have $A = I_n$.

Problem 100. Let A be an $n \times n$ matrix with identical entries:

$$A := \begin{bmatrix} a & a & \dots & a \\ a & a & \dots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & a \end{bmatrix},$$

for some $a \neq 0$. From the definition of matrix multiplication, we find that each entry of A^2 is

$$\left(A^2\right)_{ij} = \sum_{k=1}^{n} a^2 = na^2.$$

In order to have $A^2 = A$, we therefore must have

$$na^2 = a,$$

which implies $a = 1/n$. Thus the matrix

$$A := \begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n} & \dots & \frac{1}{n} \end{bmatrix},$$

satisfies $A^2 = A$. Notice A is not invertible, so there is no contradiction with problem 99.

Section 3.1

Problem 44.

(a) Since A and B are row equivalent, so are the augmented matrices $[A|\mathbf{0}]$ and $[B|\mathbf{0}]$, and so the sytems
$$A\mathbf{x} = \mathbf{0}\,, \qquad \text{and} \qquad B\mathbf{x} = \mathbf{0},$$
have the same solution sets. Thus $\ker(A) = \ker(B)$.

(b) This is not true. Here is a counterexample. Let
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
Clearly B =rref(A). Notice
$$B \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},$$
and so each vector in $\text{im}(B)$ necessarily has zero third component. This is not true of $\text{im}(A)$. For example
$$A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \notin \text{im}(B).$$

Problem 48.

(a) Since $\mathbf{w} \in \text{im}(A)$, there is some vector $\mathbf{v} \in \mathbb{R}^2$ such that $A\mathbf{v} = \mathbf{w}$. Thus $A\mathbf{w} = A(A\mathbf{v}) = A^2\mathbf{v}$. Since we assume $A^2 = A$, we have $A\mathbf{w} = A\mathbf{v} = \mathbf{w}$.

(b) If $\text{rank}(A) = 2$, then A is invertible by Theorem 2.4.3. This implies that $A = I_2$ by problem 2.4.99.

(c) If rank(A)= 1, then $\ker(A)$ contains a nonzero vector (Theorem 3.1.7), and since the kernel is closed under scalar multiplication (Theorem 3.1.6), it contains a line in the direction of this vector. Call this line L_{ker}. Similarly the image of A contains a line. In fact, the image of A is a this line. Otherwise, the image of A would include two non-parallel vectors, and therefore would include all of $\mathbb{R}^2$, by problem 2.2.33. This cannot be, as the rank of A is only 1 (see Summary 3.1.8). Denote the line of the image of A as L_{im}.

According to problem 2.2.33, if $L_{\text{im}} \neq L_{\text{ker}}$, each vector $\mathbf{v}$ in $\mathbb{R}^2$ can be uniquely expressed as
$$\mathbf{v} = \mathbf{v}_{\text{ker}} + \mathbf{v}_{\text{im}}.$$
If $L_{\text{im}} = L_{\text{ker}}$, then im($A$) =ker($A$), and so $A^2 = 0$. This is ruled out here since we assume $A^2 = A$, and A must have nonzero entries since it is assumed to have rank 1.

The projection P onto L_{im} along L_{ker} is defined in problem 2.2.33. as
$$P\mathbf{v} = P(\mathbf{v}_{\text{im}} + \mathbf{v}_{\text{ker}}) = \mathbf{v}_{\text{im}}.$$

Let $\mathbf{x} \in \mathbb{R}^2$, and write $\mathbf{x} = \mathbf{x}_{\text{im}} + \mathbf{x}_{\text{ker}}$. By linearity,

$$A\mathbf{x} = A(\mathbf{x}_{\text{im}} + \mathbf{x}_{\text{ker}}) = A\mathbf{x}_{\text{im}} + A\mathbf{x}_{\text{ker}} = A\mathbf{x}_{\text{im}},$$

since $A\mathbf{x}_{\text{ker}} = 0$. Since $\mathbf{x}_{\text{im}} \in \text{im}(A)$, there is some $\mathbf{w} \in \mathbb{R}^2$ such that $A\mathbf{w} = \mathbf{x}_{\text{im}}$, so we have

$$A\mathbf{x} = A\mathbf{x}_{\text{im}} = A(A\mathbf{w}) = A^2\mathbf{w} = A\mathbf{w} = \mathbf{x}_{\text{im}},$$

thus T is the projection P onto L_{im} along L_{ker}.

Problem 49. Let T be a linear transformation, and let $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_k} \in \ker(T)$, so that $T\mathbf{x_1} = T\mathbf{x_2} = \cdots = T\mathbf{x_k} = \mathbf{0}$. By linearity, for any scalars $a_1, a_2, \ldots a_k$,

$$T(a_1\mathbf{x_1} + a_2\mathbf{x_2} + \cdots + a_k\mathbf{x_k}) = a_1T\mathbf{x_1} + a_2T\mathbf{x_2} + \cdots + a_kT\mathbf{x_k} = a_1T\mathbf{0} + a_2T\mathbf{0} + \cdots + a_kT\mathbf{0} = \mathbf{0},$$

which implies that the linear combination $a_1\mathbf{x_1} + a_2\mathbf{x_2} + \cdots + a_k\mathbf{x_k}$ is also in the kernel of T.

Problem 50. Indeed it must be that $\ker(A^3) = \ker(A^4)$.

Proof. It is always true that $\ker(A^3) \subseteq \ker(A^4)$, since $A^3\mathbf{x} = \mathbf{0}$ implies $A^4\mathbf{x} = A(A^3\mathbf{x}) = A\mathbf{0} = \mathbf{0}$. We therefore need to show that there is no vector which is in $\ker(A^4)$ but not in $\ker(A^3)$. We proceed by contradiction. Suppose there is some vector $\mathbf{x} \in \ker(A^4) \setminus \ker(A^3)$. Then $A^4\mathbf{x} = \mathbf{0}$, and $A^3\mathbf{x} \neq \mathbf{0}$. Since $A^3(A\mathbf{x}) = A^4\mathbf{x} = \mathbf{0}$, the vector $A\mathbf{x}$ is in the kernel of A^3. Since $A^2(A\mathbf{x}) = A^3\mathbf{x} \neq \mathbf{0}$, the vector $A\mathbf{x}$ is not in the kernel of A^2. This contradicts the assumption that $\ker(A^2) = \ker(A^3)$. It follows that there is no $\mathbf{x} \in \ker(A^4) \setminus \ker(A^3)$, and thus $\ker(A^3) = \ker(A^4)$. □

Section 3.2

Problem 6. Let V and W be subspaces of $\mathbb{R}^n$.

(a) The intersection $V \cap W$ is also a subspace. To prove this we need to show that $V \cap W$ contains the zero vector, and is closed under linear combination. Since V and W be subspaces, $\mathbf{0} \in V$ and $\mathbf{0} \in W$, so clearly $\mathbf{0} \in V \cap W$.

Now consider vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in V \cap W$. Any linear combination of these vectors is in V, since the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are all in V and V is a subspace. Any linear combination of these vectors is also in W, since the vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are all in W and W is also a subspace. Thus any linear combination of the vectors $\mathbf{x}_1, x_2, \ldots, \mathbf{x}_k$ is in both V and W, thus in $V \cap W$.

(b) The union $V \cup W$ is not necessarily a subspace. Here is a counterexample. Let $V \in \mathbb{R}^2$ be the horizontal axis (x-axis), and $W \in \mathbb{R}^2$ be the vertical axis (y-axis). That is, V is spanned by the single vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and W is spanned by the single vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The union of these two axes is not closed under linear combination. For example $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin V \cup W$.

Problem 40. The columns of AB are linearly independent if and only if the linear system

$$(AB)\mathbf{x} = \mathbf{0},$$

has a unique solution. We can write the above system as

$$A(B\mathbf{x}) = \mathbf{0}.$$

Since the columns of A are linearly independent, the above equation holds if and only if $B\mathbf{x} = \mathbf{0}$. Since the columns of B are also linearly independent, $B\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Thus $(AB)\mathbf{x} = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$, so the system has a unique solution. Therefore the columns of AB are linearly independent.

Problem 42. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m \in \mathbb{R}^n$ be some unit vectors which are all perpendicular to eachother, i.e. $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$. Suppose there is some linear relation between them

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m = \mathbf{0}.$$

Consider the dot product of $\mathbf{v}_1$ with the vectors on each side of this equation. On the right side we have $\mathbf{v}_1 \cdot \mathbf{0} = 0$. On the left side we have

$$\begin{aligned}\mathbf{v}_1 \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m) &= c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) + c_2(\mathbf{v}_1 \cdot \mathbf{v}_2) + \cdots + c_m(\mathbf{v}_1 \cdot \mathbf{v}_m) \\ &= c_1||\mathbf{v}_1||^2 + c_2(0) + \cdots + c_m(0) \\ &= c_1,\end{aligned}$$

as we have assumed all of the vectors $\mathbf{v}_j$ have unit length. This implies that $c_1 = 0$. Similarly, taking the dot product of both sides with $\mathbf{v_2}$ implies that $c_2 = 0$. In general for each $j = 1, 2, \ldots, m$, taking the dot product of both sides with $\mathbf{v}_j$ implies that $c_j = 0$, and so the linear relation can only be the trivial relation. Thus the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.