MATH 217 SPRING 2014
WRITTEN HOMEWORK 6
SOLUTIONS

Section 3.1

Problem 14. Label the first, second, and third column of A as $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, respectively. Notice that $\mathbf{v}_2 = 2\mathbf{v}_1 \in \text{span}(\mathbf{v}_1)$, and $\mathbf{v}_3 = 3\mathbf{v}_1 \in \text{span}(\mathbf{v}_1)$. Thus im($A$) =span($\mathbf{v}_1$).

Problem 32. Let $T : \mathbb{R}^1 \to \mathbb{R}^3$ be defined by

$$T(x) = \begin{bmatrix} 7x \\ 6x \\ 5x \end{bmatrix}.$$

The image of T is the set of all scalar multiples of $\begin{bmatrix} 7x \\ 6x \\ 5x \end{bmatrix}$, which is the line spanned by $\begin{bmatrix} 7x \\ 6x \\ 5x \end{bmatrix}$.

Problem 42. Using the hint, we find that the system $A\mathbf{x} = \mathbf{y}$ is consistent if and only if

$$\begin{aligned} y_1 \qquad - 3y_3 \quad +2y_4 &= 0 \\ y_2 - 2y_3 \quad +y_4 &= 0. \end{aligned}$$

This is the linear system

$$B\mathbf{y} = \mathbf{0}\,, \qquad B = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -2 & 1 \end{bmatrix}.$$

A vector $\mathbf{y}$ solves this system if and only in $\mathbf{y} \in \ker(B)$, and so we have $\text{im}(A) = \ker(B)$.

Problem 43. Algorithm for writing the image of A as the kernel of some matrix: Assume A is of size $n \times m$, and consider the matrix of size $n \times (m + n)$,

$$[A|I_n]\,.$$

Use Gauss–Jordan elimination to reduce the left side of this matrix to reduced row echelon form. Suppose $\text{rank}(A) = j < n$. Then there will be $(n - j)$ rows of zeroes at the bottom of rref(A). Let B be the $(n - j) \times m$ sub-matrix of rref$[A|I_n]$ in the bottom left corner. Them im(A) =ker(B).

If $\text{rank}(A) = n$, then the system $A\mathbf{x} = \mathbf{y}$ is consistent for every $\mathbf{y} \in \mathbb{R}^n$, and so $\text{im}(A) = \mathbb{R}^n$. Then we can let B be the zero matrix of size $p \times n$, for any p. The kernel of any such matrix is $\mathbb{R}^n$.

Section 3.2

Problem 34. Since the vector $\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ is in the kernel of A, we have

$$A\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = \mathbf{0}.$$

In terms of the columns of A, this is

$$\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 4\mathbf{v}_4 = \mathbf{0},$$

which implies

$$\mathbf{v}_4 = -\frac{1}{4}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2 - \frac{3}{4}\mathbf{v}_3.$$

Problem 46. Denote $A = \begin{bmatrix}1 & 2 & 0 & 3 & 5\\0 & 0 & 1 & 4 & 6\end{bmatrix}$. Notice A is in rref. The linear system $A\mathbf{x} = \mathbf{0}$ has a solution with three free variables: $x_5 = t$, $x_4 = s$, $x_3 = -4s - 6t$, $x_2 = r$, $x_1 = -2r - 3s - 5t$, for any $t, s, r \in \mathbb{R}$. The solution set of this homogeneous sytem is thus

$$\left\{ t\begin{bmatrix}-5\\0\\-6\\0\\1\end{bmatrix} + s\begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix} + r\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} \middle| t, s, r \in \mathbb{R} \right\} = \operatorname{span}\left(\begin{bmatrix}-5\\0\\-6\\0\\1\end{bmatrix}, \begin{bmatrix}-3\\0\\-4\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} \right).$$

Notice that any multiple of the first vector has a zero in the fourth component, but the second vector does not have a zero in the fourth component, thus the first and second vectors are linearly independent. Also, any linear combination of the first two vectors has a zero in the second component, but the third vector does not have a zero in the second component. It follows that the three vectors are linearly independent, and so the form a basis for $\ker(A)$.

Problem 49. Let $A = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Then $\operatorname{im}(A)$ is the line spanned by $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

To find a matrix B such that $\ker(B)$ is this line, we can use the algorithm described in problem 43 of section 3.1. This yields

$$B = \begin{bmatrix}-1 & 1 & 0\\-1 & 0 & 1\end{bmatrix}.$$

Section 3.3

Problem 28. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 2 & 3 & 4 & k \end{bmatrix}.$$

We know that the column vectors of A form a basis for $\mathbb{R}^4$ if and only if $\mathrm{rref}(A) = I_4$. Using Gauss–Jordan elimination, we find that

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & k-4-9-16 \end{bmatrix}.$$

This matrix is row equivalent to I_4 if and only if $k - 4 - 9 - 16 \neq 0$. Thus we find the column vectors of A form a basis of $\mathbb{R}^4$ for any $k \neq 29$.

Problem 32. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$. Then $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$ if $\mathbf{x} \cdot \mathbf{v}_1 = \mathbf{x} \cdot \mathbf{v}_2 = 0$. This condition implies the linear system

$$A\mathbf{x} = \mathbf{0}\,, \qquad A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix},$$

and so we find that the set of all vectors perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$ is exactly $\ker(A)$. The matrix A is already in rref, and we see that the linear system with coefficient matrix A has two free variables, x_3 and x_4. Let $x_3 = t$ and $x_4 = s$. Then $\mathbf{x}$ solves the homogeneous system above for

$$\mathbf{x} = \begin{bmatrix} t - s \\ -2t - 3s \\ t \\ s \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix},$$

and so

$$\ker(A) = \mathrm{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right).$$

These two vectors are clearly not multiples of one another, therefore they are linearly independent and so form a basis for $\ker(A)$.

Problem 68. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 4 & 8 & 0 & 0 & 0 & 1 \end{bmatrix}$. Denote the first and second columns of this matrix by $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. The third, fourth, fifth, and sixth columns of A are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and $\mathbf{e}_4$, respectively. Using Gauss–Jordan elimination we find

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & -1 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{3}{4} \end{bmatrix}$$

The free variables are in the third and sixth columns, and so the vectors $\mathbf{e}_1$ and $\mathbf{e}_4$ are redundant vectors. It follows that $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2, \mathbf{e}_3)$ form a basis for $\mathbb{R}^4$.