MATH 217 SPRING 2014
WRITTEN HOMEWORK 7
SOLUTIONS

Section 3.3

Problem 62. Let $\dim(V) = m$ and let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V \subseteq W$ form a basis of V. hese are m linearly independent vectors in W. Theorem 3.3.4(a) implies $\dim(V) = m \leq \dim(W)$.

Problem 63. Suppose $V \subseteq W$ and $\dim(V) = \dim(W) = m$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V \subseteq W$ form a basis of V. Then they are linearly independent and $V = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. Since these are m linearly independent vectors in the m-dimensional space W, Theorem 3.3.4(c) implies they also form a basis for W and $W = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m) = V$.

Problem 66. Suppose for every $\mathbf{x} \in \mathbb{R}^n$, there exist unique $\mathbf{v} \in V$ and $\mathbf{w} \in W$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w}$.
Let $\mathbf{x} \in V \cap W$. Then $\mathbf{x} = \mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x}$ where $\mathbf{v} = \mathbf{x}, \mathbf{w} = \mathbf{0}$ or $\mathbf{v} = \mathbf{0}, \mathbf{w} = \mathbf{x}$. Uniqueness implies $\mathbf{x} = \mathbf{0}$ and $V \cap W = \{\mathbf{0}\}$.
Consider a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ of V and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_q$ of W.
Let $\mathbf{x} \in \mathbb{R}^n$. There exist $\mathbf{v} \in V = \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p)$ and $\mathbf{w} \in W = \mathrm{Span}(\mathbf{w}_1, \ldots, \mathbf{w}_q)$ such that $\mathbf{x} = \mathbf{v} + \mathbf{w} \in \mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q)$. So these $p + q$ vectors span $\mathbb{R}^n$.
Let $c_1, \ldots, c_p, d_1, \ldots, d_q \in \mathbb{R}$ such that $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p + d_1\mathbf{w}_1 + \ldots + d_q\mathbf{w}_q = \mathbf{0}$. Note that $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \in V$ and $d_1\mathbf{w}_1 + \ldots + d_q\mathbf{w}_q \in W$ and since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ is a unique representation, $c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p = 0$ and $d_1\mathbf{w}_1 + \ldots + d_q\mathbf{w}_q = \mathbf{0}$. The basis vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ of V are linearly independent, so $c_1 = \ldots = c_p = 0$. Similarly $d_1 = \ldots = d_q = 0$.
Therefore, we can see that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q$ must form a basis of $\mathbb{R}^n$ and $n = \dim(\mathbb{R}^n) = p + q = \dim(V) + \dim(W)$.

Suppose $V \cap W = \{\mathbf{0}\}$ and $\dim(V) + \dim(W) = n$.
Consider a basis $\mathbf{v}_1, \ldots, \mathbf{v}_p$ of V and a basis $\mathbf{w}_1, \ldots, \mathbf{w}_q$ of W. Above we have proved that $V \cap W = \{\mathbf{0}\}$, $\mathbf{v}_1, \ldots, \mathbf{v}_p \in V$ are linearly independent and $\mathbf{w}_1, \ldots, \mathbf{w}_q \in W$ are linearly independent implies all the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p, \mathbf{w}_1, \ldots, \mathbf{w}_q$ are linearly independent. As $\dim(V) + \dim(W) = n$, these are n linearly independent vectors in $\mathbb{R}^n$ and they must span $\mathbb{R}^n$.
So for every $\mathbf{x} \in \mathbb{R}^n$, there exist $c_1, \ldots, c_p, d_1, \ldots, d_q \in \mathbb{R}$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p \in V$, $\mathbf{w} = d_1\mathbf{w}_1 + \ldots + d_q\mathbf{w}_q \in W$ and $\mathbf{x} = \mathbf{v} + \mathbf{w}$.
Suppose there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$ such that $\mathbf{x} = \mathbf{v}_1 + \mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_2$. Then $\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{w}_2 - \mathbf{w}_1 \in V \cap W$. As $V \cap W = \{\mathbf{0}\}$, $\mathbf{v}_1 = \mathbf{v}_2$ and $\mathbf{w}_1 = \mathbf{w}_2$. Thus every $\mathbf{x} \in \mathbb{R}^n$ has a unique representation and V, W are complements in $\mathbb{R}^n$.

Section 3.4

Problem 54. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathcal{I}$ be bases of $\mathbb{R}^n$. Suppose $c_1[\mathbf{v}_1]_{\mathcal{I}} + \ldots + c_n[\mathbf{v}_n]_{\mathcal{I}} = \mathbf{0}$. Due to linearity of coordinates $[c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n]_{\mathcal{I}} = c_1[\mathbf{v}_1]_{\mathcal{I}} + \ldots + c_n[\mathbf{v}_n]_{\mathcal{I}} = \mathbf{0} = [\mathbf{0}]_{\mathcal{I}}$. Since the coordinate transformation is invertible, $c_1\mathbf{v}_1 + \ldots + c_n\mathbf{v}_n = \mathbf{0}$. $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent, therefore $c_1 = \ldots = c_n = 0$. Thus the vectors $[\mathbf{v}_1]_{\mathcal{I}}, \ldots, [\mathbf{v}_n]_{\mathcal{I}}$ are n linearly independent vectors in $\mathbb{R}^n$ and they also form a basis of $\mathbb{R}^n$.

Problem 56. Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ be the required basis of $\mathbb{R}^2$ and consider the 2×2 matrix $S = [\mathbf{v}_1 \ \mathbf{v}_2]$. For all $\mathbf{x} \in \mathbb{R}^2$, $\mathbf{x} = S[\mathbf{x}]_{\mathcal{B}}$. Therefore $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = S\begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$ and $S = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ 14 & -8 \end{bmatrix}$. That is, $\mathbf{v}_1 = \begin{bmatrix} 12 \\ 14 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -7 \\ -8 \end{bmatrix}$.

Problem 58.

(a) Consider the linear relation $c_1A^2\mathbf{v} + c_2A\mathbf{v} + c_3\mathbf{v} = \mathbf{0}$. These are all vectors in $\mathbb{R}^3$, so we can multiply both sides by A^2. Using $A^3\mathbf{v} = \mathbf{0}$, we see that $A^n\mathbf{v} = \mathbf{0}$ for all $n \geq 3$ and therefore $c_3A^2\mathbf{v} = \mathbf{0}$. Since $A^2\mathbf{v} \neq \mathbf{0}$, $c_3 = 0$.

So the linear relation must be $c_1A^2\mathbf{v} + c_2A\mathbf{v} = \mathbf{0}$. Multiplying both sides by A, we see $c_2A^2\mathbf{v} = \mathbf{0}$. So $c_2 = 0$. This means $c_1A^2\mathbf{v} = \mathbf{0}$ which implies $c_1 = 0$ as well. Any linear relation between these vectors is trivial, so they are 3 linearly independent vectors in the 3-dimensional linear space $\mathbb{R}^3$ and they must form a basis of $\mathbb{R}^3$.

(b) Let $\mathcal{B} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$ where $\mathbf{w}_1 = A^2\mathbf{v}$, $\mathbf{w}_2 = A\mathbf{v}$ and $\mathbf{w}_3 = \mathbf{v}$. We can find the $\mathcal{B}$-matrix for T column by column. The first, second and third columns of $[T]_{\mathcal{B}}$ are, respectively, $[A\mathbf{w}_1]_{\mathcal{B}}$, $[A\mathbf{w}_2]_{\mathcal{B}}$ and $[A\mathbf{w}_3]_{\mathcal{B}}$. $A\mathbf{w}_1 = A^3\mathbf{v} = \mathbf{0}$, $A\mathbf{w}_2 = A^2\mathbf{v} = \mathbf{w}_1$ and $A\mathbf{w}_3 = A\mathbf{v} = \mathbf{w}_2$. It follows that

$$[A\mathbf{w}_1]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad [A\mathbf{w}_2]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad [A\mathbf{w}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and so

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Section 4.1

Problem 46. Let $f \in W$. Then for some $a, k \in \mathbb{R}$,

$$f = (a, a+k, a+2k, a+3k, \ldots) = (a, a, a, a, \ldots) + (0, k, 2k, 3k, \ldots) = a(1, 1, 1, 1, \ldots) + k(0, 1, 2, 3, \ldots).$$

Thus $W = \text{Span}(f_1, f_2)$ where $f_1 = (1, 1, 1, 1, \ldots) \in W$ and $f_2 = (0, 1, 2, 3, \ldots) \in W$. Since $f_1 \neq 0_W$ and $f_2 \neq cf_1$ for any $c \in \mathbb{R}$, f_1, f_2 are linearly independent and form a basis of W. $\dim(W) = 2$.

Problem 49. Consider the zero transformation that maps every vector in $\mathbb{R}^m$ to $\mathbf{0} \in \mathbb{R}^n$ is clearly a linear transformation. Consider linear transformations $T_1, T_2 : \mathbb{R}^m \to \mathbb{R}^n$ and scalars $c_1, c_2 \in \mathbb{R}$. For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and scalars $a, b \in R$,

$$\begin{aligned}(c_1T_1 + c_2T_2)(a\mathbf{x} + b\mathbf{y}) &= c_1T_1(a\mathbf{x} + b\mathbf{y}) + c_2T_2(a\mathbf{x} + b\mathbf{y}) \\ &= c_1(aT_1(\mathbf{x}) + bT_1(\mathbf{y})) + c_2(aT_2(\mathbf{x}) + bT_2(\mathbf{y})) && \text{since } T_1, T_2 \text{ are linear} \\ &= a((c_1T_1 + c_2T_2)(\mathbf{x})) + b((c_1T_1 + c_2T_2)(\mathbf{y})) && \text{rearranging terms}\end{aligned}$$

We have show that $c_1T_1 + c_2T_2$ is also linear. $L(\mathbb{R}^m, \mathbb{R}^n) \subset F(\mathbb{R}^m, \mathbb{R}^n)$ contains the zero transformation and is closed under linear combinations, so it is a subspace of $F(\mathbb{R}^m, \mathbb{R}^n)$.

Problem 54. Let V be a finite-dimensional linear space and let $\dim(V) = n$. There are at most n linearly independent vectors in V. Let W be any subspace of V. Since $W \subseteq V$, there exists an integer m, such that, m is the largest number of linearly independent vectors in W. (Note that $0 \le m \le n$.)

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be a set of m linearly independent vectors in W. It is enough to show that these vectors span W, then we have a finite basis for W and $\dim(W) = m \le n$. Suppose (for the sake of contradiction) that there exists $\mathbf{x} \in W$ such that $\mathbf{x} \notin \operatorname{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_m)$. By definition of span, $\mathbf{x}$ cannot be written as a linear combination of the independent vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$. Therefore $\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{x}$ are linearly independent, but these are $m + 1$ independent vectors in W which contradicts the maximality of m. Therefore $\mathbf{v}_1, \ldots, \mathbf{v}_m$ must span W, and therefore form a basis of W. Then $\dim(W) = m \le n$.

Section 4.2

Problem 10. Let $M, N \in \mathbb{R}^{2\times 2}$ and $a, b \in \mathbb{R}$.

$$T(aM + bN) = P(aM + bN)P^{-1} = aPMP^{-1} + bPNP^{-1} = aT(M) + bT(N)$$

Therefore $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is a linear transformation.

Also $M \in \ker(T) \iff PMP^{-1} = 0 \iff M = P^{-1}0P = 0$. Since $\ker(T) = \{0\}$ and $\mathbb{R}^{2\times 2}$ is finite-dimensional, Theorem 4.2.4(c) tells us that T is an isomorphism.

Problem 54. Consider $f(t) = at^2 + bt + c \in P_2$. Then $T(f) = \frac{35}{3}a + \frac{5}{2}b + 5c$.

$$f \in \ker(T) \iff T(f) = 0 \iff c = -\frac{7}{3}a - \frac{1}{2}b \iff f(t) = a(t^2 - \frac{7}{3}) + b(t - \frac{1}{2})$$

Thus $\ker(T) = \operatorname{Span}(f_1, f_2)$ where $f_1(t) = t^2 - \frac{7}{3} \in \ker(T)$ and $f_2 = t - \frac{1}{2} \in \ker(T)$. Since f_1, f_2 are not scalar multiples of each other, they are linearly independent, they form a basis of $\ker(T)$ and $\operatorname{nullity}(T) = 2$.

Since $\dim(P_2) = 3$, the rank-nullity theorem implies $\operatorname{rank}(T) = 1$. We know $\operatorname{im}(T) \subseteq \mathbb{R}$, so $\operatorname{im}(T) = \mathbb{R}$.

Problem 56. Consider $f(t) = at^2 + bt + c \in P_2$. Then $T(f) = 2at^2 + bt$. Clearly the linearly independent function t^2 and t span $\operatorname{im}(T)$, so they form a basis of $\operatorname{im}(T)$ and $\operatorname{rank}(T) = 2$. Also $T(f) = 0 \iff a = b = 0 \iff T(f) = c$. Then the constant function 1 forms a basis of $\ker(T)$ and $\operatorname{nullity}(T) = 1$.

Problem 68. Consider $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $T(M) = \begin{bmatrix} 3a & -b \\ (5-k)c & (1-k)d \end{bmatrix}$. $T(M) = 0$ if and only if $a = b = 0$ and $(5-k)c = 0$ and $(1-k)d = 0$. If $k = 5$, then $c \neq 0$ is possible and $\ker(T) \neq \{0\}$. Similarly, if $k = 1$, then $d \neq 0$ is possible and $\ker(T) \neq \{0\}$. However, if $k \neq 1$ and $k \neq 5$, $T(M) = 0 \iff a = b = c = d = 0 \iff M = 0$ and $\ker(T) = \{0\}$.

Problem 76. From our study of invertible functions, we know that if T is an invertible function from a set V to a set W, then its inverse $T^{-1} : W \to V$ is also an invertible function. If T is an isomorphism (an invertible linear transformation) from a linear space V to a linear space W, then we only need to prove $T^{-1} : W \to V$ is also a linear transformation.
Consider $f, g \in W$ and $a, b \in \mathbb{R}$. Note that $f = T(T^{-1}(f))$ and $g = T(T^{-1}(g))$.

$$\begin{aligned} af + bg &= aT(T^{-1}(f)) + bT(T^{-1}(g)) \\ &= T(aT^{-1}(f) + bT^{-1}(g)) \qquad \text{since } T \text{ is linear.} \\ \text{Then} \quad T^{-1}(af + bg) &= aT^{-1}(f) + bT^{-1}(g) \end{aligned}$$

Problem 77. Let $T : V \to W$ and $L : W \to U$ be isomorphisms, or invertible linear transformations.

Consider $f, g \in V$ and $a, b \in \mathbb{R}$.

$$\begin{aligned} L \circ T(af + bg) &= L(T(af + bg)) \\ &= L(aT(f) + bT(g)) && \text{since } T \text{ is linear} \\ &= aL(T(f)) + bL(T(g)) && \text{since } L \text{ is linear} \\ &= a(L \circ T)(f) + b(L \circ T)(g) \end{aligned}$$

Thus, $L \circ T : V \to U$ is linear.

To show $L \circ T$ is invertible as well, consider $g \in U$ and the equation $(L \circ T)(f) = g$.

$$\begin{aligned} L(T(f)) &= g \\ \iff \quad T(f) &= L^{-1}(g) && \text{since } L \text{ is invertible} \\ \iff \quad f &= T^{-1}(L^{-1}(g)) && \text{since } T \text{ is invertible.} \end{aligned}$$

For every $g \in U$, the equation $(L \circ T)(f) = g$ has the unique solution $f = (T^{-1} \circ L^{-1})(g)$. So the linear transformation $L \circ T$ is invertible, that is, $L \circ T$ is an isomorphism.