MATH 217 SPRING 2014
WRITTEN HOMEWORK 8
SOLUTIONS

Problem 1. Let $T : V \to W$ be an invertible linear transformation.
Suppose $f_1, \ldots, f_m \in V$ are linearly independent. Consider any linear relation

$$\begin{array}{rrcll} & c_1T(f_1) + \ldots + c_mT(f_m) & = & 0_W & \\ \iff & T(c_1f_1 + \ldots c_mf_m) & = & T(0_V) & \text{since } T \text{ is linear} \\ \iff & c_1f_1 + \ldots c_mf_m & = & 0_V & \text{since } T \text{ is invertible} \end{array}$$

Only the trivial relation exists between $f_1, \ldots, f_m$, so $c_1 = \ldots = c_m = 0$. This implies $T(f_1), \ldots, T(f_m) \in W$ are linearly independent.

Note that $T^{-1} : W \to V$ is also an isomorphism and $T^{-1}(T(f)) = f$ for any $f \in V$. We can apply our result above to T^{-1} to prove the converse.

Problem 2. Let $\mathcal{B}$ be a basis of an n-dimensional linear space V. The coordinate transformation $L_{\mathcal{B}} : V \to \mathbb{R}^n$ is an isomorphism. Using the result of Problem 1, we can see that $f_1, \ldots, f_m \in V$ are linearly independent if and only if $[f_1]_{\mathcal{B}}, \ldots, [f_m]_{\mathcal{B}} \in \mathbb{R}^n$ are linearly independent.

$$\begin{array}{rrcll} & f & = & c_1f_1 + \ldots + c_mf_m & \\ \iff & L_{\mathcal{B}}(f) & = & L_{\mathcal{B}}(c_1f_1 + \ldots + c_mf_m) & \\ \iff & [f]_{\mathcal{B}} & = & c_1[f_1]_{\mathcal{B}} + \ldots + c_m[f_m]_{\mathcal{B}} & \text{since } L_{\mathcal{B}} \text{ is linear.} \end{array}$$

Therefore $f \in \text{Span}(f_1, \ldots, f_m)$ if and only if $[f]_{\mathcal{B}} \in \text{Span}([f_1]_{\mathcal{B}}, \ldots, [f_m]_{\mathcal{B}})$.

(a) Now what remains to be shown is that $\ker(T) = \text{Span}(f_1, \ldots, f_m)$ if and only $\ker([T]_{\mathcal{B}}) = \text{Span}([f_1]_{\mathcal{B}}, \ldots, [f_m]_{\mathcal{B}})$ or simply, $f \in \ker(T)$ if and only if $[f]_{\mathcal{B}} \in \ker([T]_{\mathcal{B}})$.

$$\begin{array}{rll} & f \in \ker(T) & \\ \iff & T(f) = 0_V & \\ \iff & [T(f)]_{\mathcal{B}} = \mathbf{0} & \text{since } L_{\mathcal{B}} \text{ is an isomorphism} \\ \iff & [T]_{\mathcal{B}}[f]_{\mathcal{B}} = \mathbf{0} & \text{using the characterizing equation of the } \mathcal{B}\text{-matrix} \\ \iff & [f]_{\mathcal{B}} \in \ker([T]_{\mathcal{B}}) & \end{array}$$

(b) Similarly

$$
\begin{array}{lll}
 & g \in \operatorname{im}(T) & \\
\iff & T(f) = g & \text{for some } f \in V \\
\iff & [T(f)]_{\mathcal{B}} = [g]_{\mathcal{B}} & \text{since } L_{\mathcal{B}} \text{ is an isomorphism} \\
\iff & [T]_{\mathcal{B}}[f]_{\mathcal{B}} = [g]_{\mathcal{B}} & \text{using the characterizing equation of the } \mathcal{B}\text{-matrix} \\
\iff & [g]_{\mathcal{B}} \in \operatorname{im}([T]_{\mathcal{B}}) &
\end{array}
$$

Section 4.3

Problem 59. Define $T : P \to P$ such that $T(f(x)) = \int_0^x f(t)\,dt$. Since integration respects linearity, you can verify that for all $f, g \in P_2$ and $a, b \in \mathbb{R}$, $T(af + bg) = aT(f) + bT(g)$ and T is linear. Note that the constant term in $T(f)$ must equal 0. If $f \neq 0$, then $T(f) \neq 0$; in fact the degree of $T(f)$ must be at least 1. For example, $1 \notin \operatorname{im}(T)$. So $\ker(T) = \{0\}$ but $\operatorname{im}(T) \neq P$.

Problem 69. Let M such that $m_{ij} = f_j(a_i)$. Then $\mathbf{c} \in \ker(M)$ if and only if $M\mathbf{c} = \mathbf{0}$ if and only if for each i, $\sum_{j=1}^{n} m_{ij}c_j = 0$ that is $\sum_{j=1}^{n} c_j f_j(a_i) = 0$.

This implies the polynomial $f = \sum_{j=1}^{n} c_j f_j \in P_{n-1}$ has n distinct roots $a_1, \ldots, a_n$. This violates the Fundamental Theorem of Algebra, unless $f = 0$. As basis elements $f_1, \ldots, f_n$ are linearly independent, $\sum_{j=1}^{n} c_j f_j = 0$ implies $c_1 = \ldots = c_n = 0$, that is, $\mathbf{c} = \mathbf{0}$. This shows $\ker(M) = \{\mathbf{0}\}$ and the $n \times n$ matrix M must be invertible.

Section 5.1

Problem 12.

$$
\begin{array}{rcll}
\|\mathbf{v} + \mathbf{w}\|^2 & = & (\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} + \mathbf{w}) & \\
 & = & \mathbf{v} \cdot \mathbf{v} + 2(\mathbf{v} \cdot \mathbf{w}) + \mathbf{w} \cdot \mathbf{w} & \\
 & \leq & \|\mathbf{v}\|^2 + 2|\mathbf{v} \cdot \mathbf{w}| + \|\mathbf{w}\|^2 & \\
 & \leq & \|\mathbf{v}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| + \|\mathbf{w}\|^2 & \text{using Cauchy-Schwarz inequality} \\
\|\mathbf{v} + \mathbf{w}\| & \leq & \|\mathbf{v}\| + \|\mathbf{w}\| & \text{taking the positive square root on both sides}
\end{array}
$$

Problem 22. Suppose $\mathbf{x}$ is orthogonal to each basis vector $\mathbf{v}_1, \ldots, \mathbf{v}_m$ of V, that is, $\mathbf{x} \cdot \mathbf{v}_i = 0$ for each i. Let $\mathbf{v} \in V$. There exist scalars $c_1, \ldots, c_m$ such that $\mathbf{v} = c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m$. Then $\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot (c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m) = c_1(\mathbf{x} \cdot \mathbf{v}_1) + \ldots + c_m(\mathbf{x} \cdot \mathbf{v}_m) = 0$. $\mathbf{x}$ is orthogonal to any $\mathbf{v} \in V$, and hence to the subspace V.

The converse is obviously true since $\mathbf{v}_1, \ldots, \mathbf{v}_m \in V$.

Problem 23. Let $\mathbf{v} \in V$. By definition of $V^\perp$, $\mathbf{v} \cdot \mathbf{w} = 0$ for any $\mathbf{w} \in V^\perp$. Therefore $\mathbf{v} \in (V^\perp)^\perp$ and $V \subseteq (V^\perp)^\perp$, which implies $\dim(V) \leq \dim((V^\perp)^\perp)$.

Using Theorem 5.1.8(c) twice, $\dim(V) = n - \dim(V^\perp) = \dim((V^\perp)^\perp)$. As $V \subseteq (V^\perp)^\perp$ and $\dim(V) = \dim((V^\perp)^\perp) \leq n$, we see that $V = (V^\perp)^\perp$

Problem 25.

(a) $\|k\mathbf{v}\| = \sqrt{(k\mathbf{v})\cdot(k\mathbf{v})} = \sqrt{k^2(\mathbf{v}\cdot\mathbf{v})} = \sqrt{k^2}\sqrt{\mathbf{v}\cdot\mathbf{v}} = |k|\|\mathbf{v}\|$.

(b) Since $\|\mathbf{v}\| \neq 0$, let $k = \dfrac{1}{\|\mathbf{v}\|}$ in part (a). Then $\|\mathbf{u}\| = \dfrac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1$.