MATH 217 SPRING 2014
WRITTEN HOMEWORK 9
SOLUTIONS

Section 4.3

Problem 50. Let's find $[T]_{\mathcal{B}}$, the matrix for T in the basis $\mathcal{B} = (\cos(t), \sin(t))$. We have

$$T(\cos(t)) = (b-1)\cos(t) - a\sin(t)\,, \qquad T(\cos(t)) = a\cos(t) + (b-1)\sin(t)\,,$$

and so

$$[T(\cos(t))]_{\mathcal{B}} = \begin{bmatrix} b-1 \\ -a \end{bmatrix} \qquad [T(\sin(t))]_{\mathcal{B}} = \begin{bmatrix} a \\ b-1 \end{bmatrix}, \qquad [T]_{\mathcal{B}} = \begin{bmatrix} b-1 & a \\ -a & b-1 \end{bmatrix}.$$

T is an isomorphism if and only if this matrix is invertible. The determinant of $[T]_{\mathcal{B}}$ is $a^2 + (b-1)^2$, which is zero if and only if $a = 0$ and $b = 1$. Thus T is an isomorphism for all values of a, b except $a = 0, b = 1$.

Problem 60.

(a) Notice that $\mathbf{b}_1 = \mathbf{a}_1$, and $\mathbf{b}_2 = \mathbf{a}_1 + \mathbf{a}_2$ Thus $[\mathbf{b}_1]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $[\mathbf{b}_2]_{\mathcal{U}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The change of basis matrix from $\mathcal{B}$ to $\mathcal{U}$ is therefore

$$S_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

(b) Since for any $\mathbf{x} \in V$, $[\mathbf{x}]_{\mathcal{U}} = S_{\mathcal{B}\to\mathcal{U}}[\mathbf{x}]_{\mathcal{B}}$, we have $S^{-1}_{\mathcal{B}\to\mathcal{U}}[\mathbf{x}]_{\mathcal{U}} = [\mathbf{x}]_{\mathcal{B}}$. Thus the change of basis matrix from $\mathcal{U}$ to $\mathcal{B}$ is

$$S_{\mathcal{U}\to\mathcal{B}} = S^{-1}_{\mathcal{B}\to\mathcal{U}} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

(c) They are related by

$$\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix} S_{\mathcal{B}\to\mathcal{U}},$$

which is equivalent to the equations $\mathbf{b}_1 = \mathbf{a}_1$, and $\mathbf{b}_2 = \mathbf{a}_1 + \mathbf{a}_2$.

Problem 64. Denote

$$m_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so that the basis $\mathcal{U}$ is $\mathcal{U} = (m_1, m_2, m_3)$. Notice that

$$I_2 = m_1 + m_3, \qquad Q = m_1 + 2m_2 + 3m_3, \qquad Q^2 = \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} = m_1 + 8m_2 + 9m_3.$$

(a) We have

$$T(m_1) = I_2 = m_1 + m_3, \quad T(m_2) = Q = m_1 + 2m_2 + 3m, \quad T(m_3) = m_1 + 8m_2 + 9m_3.$$

The matrix for T in the basis $\mathcal{U}$ is thus

$$[T]_{\mathcal{U}} = \begin{bmatrix} | & | & | \\ [Tm_1]_{\mathcal{U}} & [Tm_2]_{\mathcal{U}} & [Tm_3]_{\mathcal{U}} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 1 & 3 & 9 \end{bmatrix}.$$

(b) Let $m = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ be an arbitrary matrix in V. Then

$$Tm = aI_2 + bQ + cQ^2 = \begin{bmatrix} a+b+c & 2b+8c \\ 0 & a+3b+9c \end{bmatrix} = a(m_1+m_3)+b(m_1+2m_2+3m_3)+c(m_1+8m_2+9m_3).$$

Since $a, b,$ and c can be any real numbers, we see that $\mathrm{im}(T) = \mathrm{span}(m_1+m_3, m_1+2m_2+3m_3, m_1+8m_2+9m_3)$. Notice that $-3(m_1+m_3)+4(m_1+2m_2+3m_3) = m_1+8m_2+9m_3$, and so the element $m_1+8m_2+9m_3$ is a redundant one, and $\mathrm{im}(T) = \mathrm{span}(m_1+m_3, m_1+2m_2+3m_3)$. These two elements are linearly independent, as only one of them has a m_2 term, and thus form a basis for $\mathrm{im}(T)$. Since $\dim(\mathrm{im}(T)) = 2$, we have $\mathrm{rank}(T) = 2$.

By the rank-nullity theorem, the dimension of the kernel of T is 1, so to find a basis for the kernel it suffices to find a single nonzero element of the kernel. Using the relation described above, $-3(m_1+m_3)+4(m_1+2m_2+3m_3)-(m_1+8m_2+9m_3) = 0$, we find that $\begin{bmatrix} -3 & 4 \\ 0 & -1 \end{bmatrix} \in \ker(T)$, and so

$$\ker(T) = \mathrm{span}\left(\begin{bmatrix} -3 & 4 \\ 0 & -1 \end{bmatrix}\right).$$

Section 5.1

Problem 16. Let $\mathbf{u}_4 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$. If $\mathbf{u}_4$ is orthogonal to $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, then

$$\begin{aligned} \frac{x_1}{2} + \frac{x_2}{2} + \frac{x_3}{2} + \frac{x_4}{2} &= 0 \\ \frac{x_1}{2} + \frac{x_2}{2} - \frac{x_3}{2} - \frac{x_4}{2} &= 0 \\ \frac{x_1}{2} - \frac{x_2}{2} + \frac{x_3}{2} - \frac{x_4}{2} &= 0, \end{aligned}$$

which is the linear system

$$A\mathbf{u}_4 = \mathbf{0}, \qquad A = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}.$$

Notice that the rows of A are the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, i.e.

$$A = \begin{bmatrix} - & \mathbf{u}_1 & - \\ - & \mathbf{u}_2 & - \\ - & \mathbf{u}_3 & - \end{bmatrix}.$$

Using Gauss–Jordan elimination we find

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

which implies

$$\mathbf{u}_4 = \begin{bmatrix} t \\ -t \\ -t \\ t \end{bmatrix}$$

where t is a free parameter. If $\mathbf{u}_4$ is to be a unit vector, we must have $4t^2 = 1$, which implies $t = \pm\frac{1}{2}$. We therefore find there are two possibilities:

$$\mathbf{u}_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \qquad \text{or} \qquad \mathbf{u}_4 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Problem 28. Since the vectors $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ are already mutually orthogonal, we can obtain an orthonormal basis for their span simply by dividing each of them by their length. We thus obtain the orthonormal basis for the span of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$:

$$\mathcal{U} = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3), \qquad \mathbf{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}, \ \mathbf{u}_3 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}.$$

Call the span of these vectors V. The orthogonal projection of $\mathbf{e}_1$ onto V is

$$\operatorname{proj}_V \mathbf{e}_1 = (\mathbf{e}_1 \cdot \mathbf{u}_1)\mathbf{u}_1 + (\mathbf{e}_1 \cdot \mathbf{u}_2)\mathbf{u}_2 + (\mathbf{e}_1 \cdot \mathbf{u}_3)\mathbf{u}_3 = \frac{1}{2}(\mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3) = \frac{1}{4}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}.$$

Section 5.2

Problem 34. To find the kernel of $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, we first put it in rref: $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$. It follows that $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \ker(A)$ if and only if $\mathbf{x} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$, where x_3 and x_4 are free variables. Thus

$$\ker(A) = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2), \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\mathbf{v}_1, \mathbf{v}_2$ form a basis for $\ker(A)$. To find an orthonormal basis we use the Gram-Schmidt process on these vectors, to obtain the vectors $\mathbf{u}_1, \mathbf{u}_2$, where

$$\mathbf{u}_1 = \frac{1}{||\mathbf{v}_1||}\mathbf{v}_1\,, \qquad \mathbf{u}_2 = \frac{1}{||\mathbf{v}_2^\perp||}\mathbf{v}_2^\perp\,, \qquad \mathbf{v}_2^\perp = \mathbf{v}_2 - (\mathbf{u}_1 \cdot \mathbf{v}_2)\mathbf{u}_1\,.$$

$\mathbf{u}_1$ is straightforward to compute: $\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$. We can then compute $\mathbf{v}_2^\perp$:

$$\mathbf{v}_2^\perp = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{6}}(2+6+0+0)\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}.$$

Finally we have

$$\mathbf{u}_2 = \frac{1}{||\mathbf{v}_2^\perp||}\mathbf{v}_2^\perp = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}.$$

Thus the vectors

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{u}_2 = \frac{1}{\sqrt{30}} \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$$

form an orthonormal basis for $\ker(A)$.

Problem 36. This is almost already the QR-factorization. The only problem is that one of the diagonal elements in the upper-triangular factor is not positive. To change it to a positive number, consider each column of the matrix M separately. We have

$$M = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix}$$

where

$$\mathbf{v}_1 = 2\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{v}_2 = 3\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - 4\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad \mathbf{v}_3 = 5\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 6\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} + 7\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

Notice that $\mathbf{v}_2$ and $\mathbf{v}_3$ can be rewritten as

$$\mathbf{v}_2 = 3\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + 4\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}, \quad \mathbf{v}_3 = 5\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - 6\begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix} + 7\begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix},$$

and so M can be factored as

$$M = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 4 & -6 \\ 0 & 0 & 7 \end{bmatrix},$$

which is the QR-factorization.

Section 5.3

Problem 34. The dot product of the first and third columns is c and the dot product of the second and third columns is d, and so if the columns are to be orthonormal we must have $c = d = 0$. Since the first and second columns must be orthonomal, we arrive at the nonlinear equations

$$ab + ef = 0, \quad a^2 + e^2 = 1, \quad b^2 + f^2 = 1.$$

We also know that the first and third rows must be orthonormal, which gives the equations

$$ae + bf = 0, \quad a^2 + b^2 = 1, \quad e^2 + f^2 = 1.$$

In particular, combining the equations $a^2 + e^2 = 1$ and $a^2 + b^2 = 1$ gives $b^2 = f^2$, and combining the equations $b^2 + f^2 = 1$ and $a^2 + b^2 = 1$ gives $a^2 = f^2$, thus $e = \pm b$ and $f = \pm b$. Since $ab + ef = 0$, the only possibilities are $e = b, f = -a$, or $e = -b, f = a$. We thus find that the matrix must be either

$$\begin{bmatrix} a & b & 0 \\ 0 & 0 & 1 \\ b & -a & 0 \end{bmatrix}, \quad a, b \in \mathbb{R},$$

or

$$\begin{bmatrix} a & b & 0 \\ 0 & 0 & 1 \\ -b & a & 0 \end{bmatrix}, \quad a, b \in \mathbb{R}.$$

In each case, either a or b may be zero, but not both.

Problem 50.

(a) Consider an $n \times n$ upper triangular matrix A, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ denote its columns. Since A is upper triangular, $\mathbf{v}_1 = a_{11}\mathbf{e}_1$, where $\mathbf{e}_1$ is the first standard vector and $a_{11} > 0$. Since A is orthogonal, $||\mathbf{v}_1|| = 1$, which implies that $a_{11} = 1$, and so $\mathbf{v}_1 = \mathbf{e}_1$. Now consider the second column, $\mathbf{v}_2$. Since A is upper triangular, $\mathbf{v}_2 = a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2$, where $a_{12} \in \mathbb{R}$ and $a_{22} > 0$. The condition $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$ implies that $a_{12} = 0$, and the condition $||\mathbf{v}_2|| = 1$ implies $a_{22} = 1$, thus $\mathbf{v}_2 = \mathbf{e}_2$. Similarly, we find $\mathbf{v}_k = \mathbf{e}_k$ for each $k = 1, 2, \ldots, n$, and so A must be the identity matrix, I_n.

(b) Let B be an invertible matrix of size $n \times n$, and suppose we can factor B in the ways $B = Q_1R_1$ and $B = Q_2R_2$, where Q_1 and Q_2 are orthogonal matrices, and R_1 and R_2 are upper triangular with positive entries on the diagonal. In particular, all of the matrices Q_1, Q_2, R_1 and R_2 are invertible, and we have the equation $Q_1R_1 = Q_2R_2$, which implies $Q_2^{-1}Q_1 = R_2R_1^{-1}$. Denote $C = Q_2^{-1}Q_1 = R_2R_1^{-1}$. Since C is the product of the orthogonal matrices, it is orthogonal. Since C is the product of upper triangular matrices, it is orthogonal as well. According to part (a) then, we must have $C = I_n$. Thus $Q_2^{-1}Q_1 = I_n$ and $R_2R_1^{-1} = I_n$, and so $Q_1 = Q_2$ and $R_1 = R_2$. Thus there is a unique QR-factorization for B.

Problem 60. Denote

$$m_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad m_4 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and denote the basis $\mathcal{B} = (m_1, m_2, m_3, m_4)$. We have $L(m_1) = m_1$, $L(m_2) = m_2$, $L(m_3) = m_3$, and $L(m_4) = -m_4$. The matrix for the transformation L in the basis $\mathcal{B}$ is thus

$$[L]_{\mathcal{B}} = \begin{bmatrix} | & | & | & | \\ [L(m_1)]_{\mathcal{B}} & [L(m_2)]_{\mathcal{B}} & [L(m_3)]_{\mathcal{B}} & [L(m_4)]_{\mathcal{B}} \\ | & | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$