MATH 255 **Applied Honors Calculus III** **Winter 2011**

Homework 1

Section 11.1, pg. 692: 8, 24, 43.
Section 11.2, pg. 702: 10, 20 (no graph required), 32, 40.
Section 11.3, pg. 713: 14, 20, 54, 81.
Section 11.4, pg. 719: 6, 35, 46.

Solutions

11.1: #8

a) First generate a table of values:

Table 1: 11.1:8

t	x	y
-2	-5	-2
-1	-2	1
0	1	2
1	4	1
2	7	-2

Then plot the values (see Figure 1).

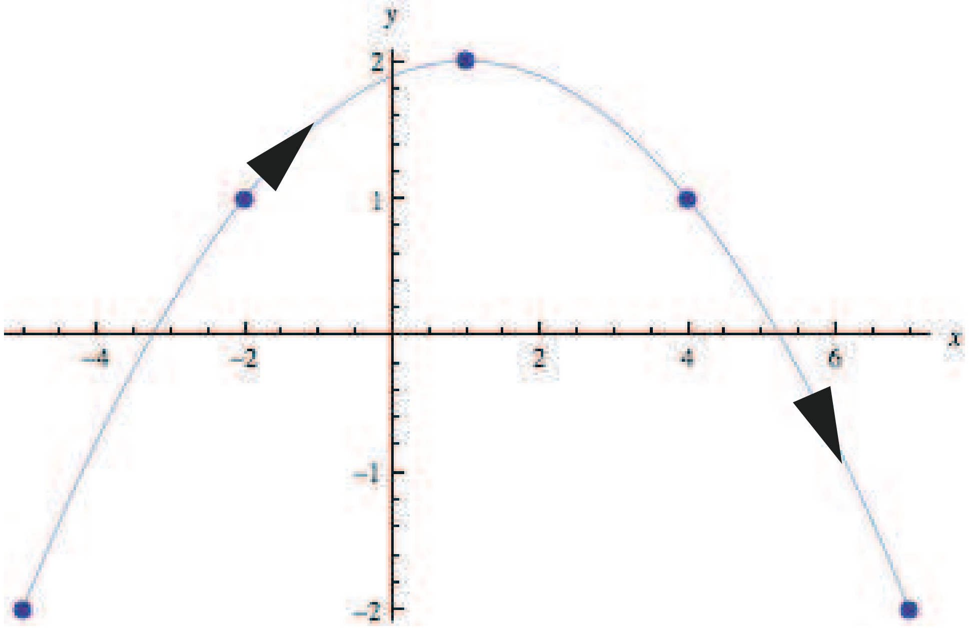

Figure 1: Parametric plot for 11.1 #8.

b) Solve $x = 1 + 3t$ for t, $t = \frac{x-1}{3}$, and plug into $y = 2 - t^2$. You get the Cartesian equation $y = \frac{17}{9} - \frac{x^2}{9} + \frac{2x}{9}$.

11.1: #24

a - III : x ranges from 1 to 2

b - I : x changes directions 4 times; 6 changes directions 6 times

c - IV : y is always positive

d - II : y is almost constant for a span, and x is as well.

11.1: #43

a) This is easiest if you first plot all the functions against t (see fig 2).

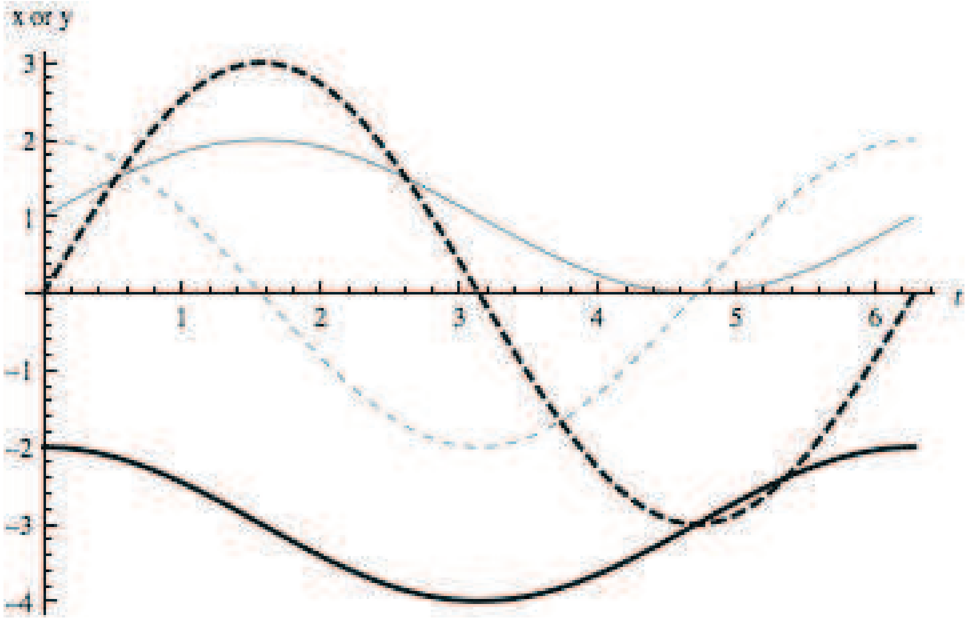

Figure 2: Graph of parametric equations vs t for 11.1 #43. Dashed = 1st particle, Solid = 2nd particle, thick=x, thin=y.

Now we transfer this information to the to the Cartesian plane for the plot of the particle trajectories (see fig 3). We could note from experience that the first particle traces an ellipse centered at the origin, and the second traces a circle of radius 1 centered at $(-3, 1)$.

b) We see two spots where the trajectories cross, but these are only collisions if the x coordinates intersect at the same time the y coordinates do. Looking again at the first plot, we see that this happens only for $t = \frac{3\pi}{2}$, which corresponds to the Cartesian point $(-3, 0)$.

c) The second particle is shifted 6 units to the right. There are still two intersection points, but no collision points. Replot the parametric equations to verify.

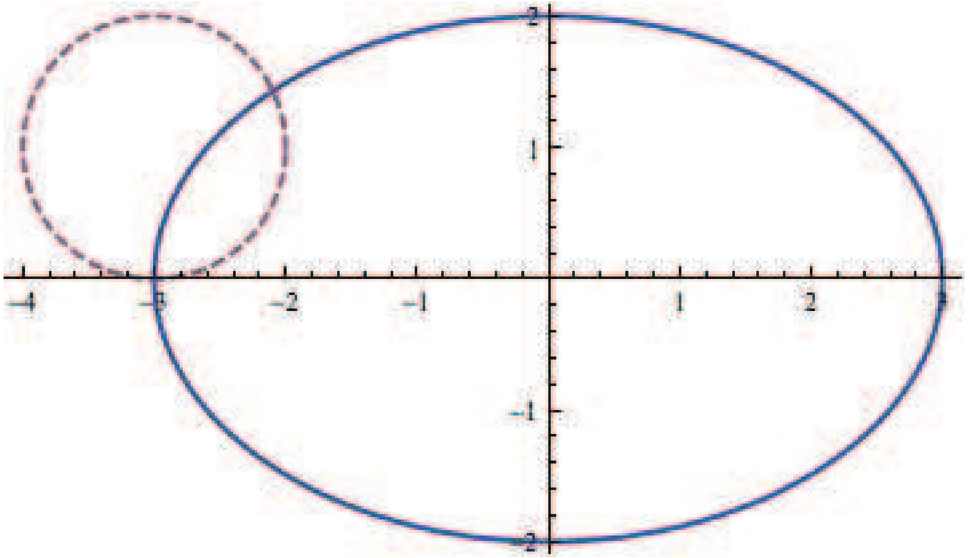

Figure 3: Parametric plot for 11.1 #43. There are two intersections, which are potential collision points.

11.2: #10

$\frac{dx}{dt} = \cos(t)$
$\frac{dy}{dt} = \cos(t + \sin(t)) \cdot (1 + \cos(t))$
$\frac{dy}{dx} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{\cos(t+sin(t))\cdot(1+\cos(t))}{\cos(t)}$

Now find the values of t for which x = 0 and y = 0:
First $x = \sin t = 0$
implies $t = 0$ and π in one cycle.

As for $y = \sin(t + \sin(t)) = 0$, it is hard to find the root, so we test the values from above: 0 and π, and we find both values work. So $t = 0, \pi$ in one cycle. Therefore, $(x, y) = (0, 0)$ when $t = 0, \pi$. Plug these values into the equation for the tangent and get $\frac{dy}{dx} = 2$ when $t = 0$, and $\frac{dy}{dx} = 0$ when $t = \pi$. This can be clearly seen from Fig. 4.

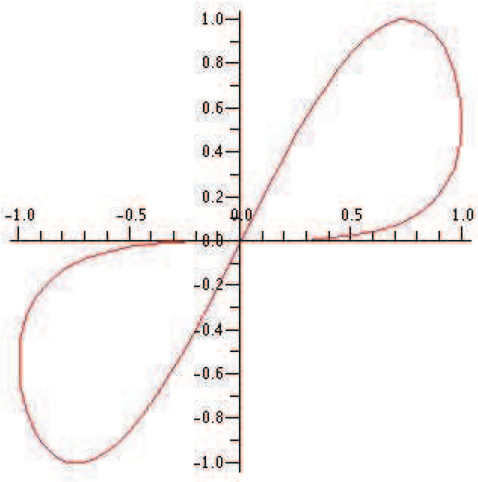

Figure 4: Graph of parametric curve $(\sin(t), \sin(t + \sin(t)))$ for 11.2 #10.

11.2: #20

We have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos\theta}{-3\sin 3\theta}.$$

Let's check for which values of θ the numerator vanishes. Clearly, in one periodic cycle, we have $2\cos\theta = 0$ if $\theta = \pi/2, 3\pi/2$.
What about the denominator? We have
$\frac{dx}{d\theta} = -3\sin 3\theta = 0$ implies $3\theta = 0, \pi$ in one cycle, so $3\theta = k\pi$ for any integer k, so $\theta = \frac{k\pi}{3}$. Therefore in one cycle, the denominator vanishes for $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$, six values

in one cycle.

Notice the roots for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ do not overlap.

When the tangent is horizontal, $\frac{dy}{dx} = 0$, i.e. when $\frac{dy}{d\theta} = 0$, but $\frac{dx}{d\theta} \neq 0$, then we get $\theta = \pi/2, 3\pi/2$ in one cycle. These correspond to the Cartesian points $(0, 2)$ and $(0, -2)$.

When it is vertical, $\frac{dy}{dx}$ goes to infinity, i.e. when $\frac{dx}{d\theta} = 0$, but $\frac{dy}{d\theta} \neq 0$, we get $\theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$ in one cycle. These correspond to the Cartesian points $(1, 0)$, $(-1, \sqrt{3})$, $(1, \sqrt{3})$, $(-1, 0)$, $(1, -\sqrt{3})$, $(-1, -\sqrt{3})$, respectively.

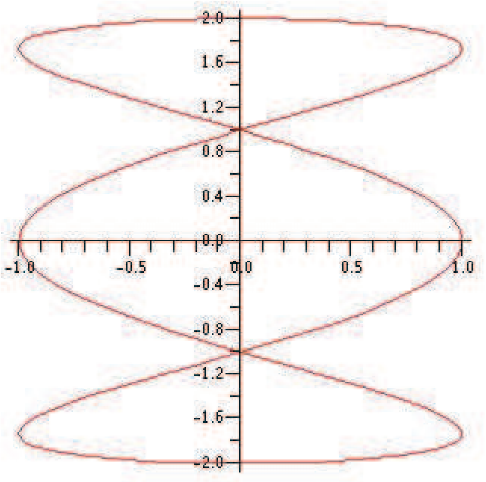

Figure 5: Graph of parametric curve $(\cos(3t), 2\sin(t))$ for 11.2 #20.

11.2: #32

Set $t + 1/t = 2.5$ to obtain $t_1 = 0.5$ and $t_2 = 2$, plug them into $x = t - 1/t$ to get $x_1 = -1.5$ and $x_2 = 1.5$. the area bounded can be calculated by:

$$\begin{aligned} A = \int_{0.5}^{2} (2.5 - y(t))x'(t)dt &= \int_{0.5}^{2} \left(2.5 - t - \frac{1}{t}\right)\left(1 + \frac{1}{t^2}\right) dt \\ &= \int_{0.5}^{2} \left(2.5 - t - \frac{2}{t} + \frac{2.5}{t^2} - \frac{1}{t^3}\right) dt \\ &= \left(2.5t - \frac{t^2}{2} - 2\ln t - \frac{2.5}{t} + \frac{1}{2t^2}\right)\Bigg|_{0.5}^{2} = 3.75 - 4\ln 2 \approx 9.8. \end{aligned}$$

11.2: #40

$\frac{dx}{dt} = 1/t$

$\frac{dy}{dt} = \frac{1}{2\sqrt{t+1}}$

$L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2} dt$
$L = \int_1^5 \sqrt{1/t^2 + \frac{1}{4(t+1)}} dt$

11.3: #14
Start with the definition of distance in Cartesian coordinates:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Make the substitutions for $x = r \cdot \cos\theta$ and $y = r \cdot \sin\theta$ and distribute.

$$D = \sqrt{(r_1 \cos\theta_1 - r_2 \cos\theta_2)^2 + (r_1 \sin\theta_1 - r_2 \sin\theta_2)^2}$$

$$D = \sqrt{r_1^2 \cos^2\theta_1 + r_2^2 \cos^2\theta_2 - 2r_1 r_2 \cos\theta_1 \cos\theta_2 + r_1^2 \sin^2\theta_1 + r_2^2 \sin^2\theta_2 - 2r_1 r_2 \sin\theta_1 \sin\theta_2}$$

$$D = \sqrt{r_1^2(\cos^2\theta_1 + \sin^2\theta_1) + r_2^2(\cos^2\theta_2 + \sin^2\theta_2) - 2r_1 r_2 \cos\theta_1 \cos\theta_2 - 2r_1 r_2 \sin\theta_1 \sin\theta_2}$$

Finally, use the sum-angle identity $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$.

$$D = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}.$$

11.3: #20

$$\begin{aligned} r &= \tan\theta \sec\theta \\ r &= \frac{sin\theta}{\cos^2\theta} \\ r\cos\theta &= \tan\theta \\ x &= \frac{y}{x} \\ y &= x^2. \end{aligned}$$

11.3: #54
(a): VI, In general, there are 2 points on the curve for each angle θ.
(b): III, In general, there are 2 points on the curve for each angle θ.
(c): IV, r is infinite for $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5\pi}{6}$.
(d): V, r is an even function of θ thus the graph is symmetric about x-axis. Furthermore, the graph is unbounded.
(e): II, $|r$ is maximal when $\theta = 0, \pm\frac{\pi}{5}, \pm\frac{2\pi}{5}$.
(f): I, r becomes smaller and smaller number as θ increases from 0.

11.3: #81
The slope of the tangent is given by:

$$\tan\phi = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\sin\theta + r/r'\cos\theta}{\cos\theta - r/r'\sin\theta} = \frac{\tan\theta + r/r'}{1 - r/r'\tan\theta}.$$

using the notation $r' = \frac{dr}{d\theta}$.

Since $\psi = \phi - \theta$, $\tan\psi = \tan(\phi - \theta)$,

$$\tan\psi = \frac{\tan\phi - \tan\theta}{1 + \tan\phi \cdot \tan\theta} = \frac{\frac{\tan\theta + r/r'}{1 - r/r'\tan\theta} - \tan\theta}{1 + \tan\theta\frac{\tan\theta + r/r'}{1 - r/r'\tan\theta}}$$

Consider just the numerator of this expression: combine the two terms by finding a common denominator. Do this for the fraction in the denominator as well.

$$\tan\psi \quad = \quad \frac{\tan\theta + r/r' - \tan\theta - r/r'\tan^2\theta}{1 - r/r'\tan\theta + \tan^2\theta + r/r'\tan\theta} = r/r'\frac{1 + \tan^2\theta}{1 + \tan^2\theta} = \frac{r}{r'}.$$

11.4: #6

$$A = \frac{1}{2}\int_{\pi/2}^{\pi}(1+\sin\theta)^2 d\theta = \frac{1}{2}\int_{\pi/2}^{\pi}(1+2\sin\theta+\sin^2\theta)d\theta$$
$$= \frac{1}{2}\int_{\pi/2}^{\pi}\left(1+2\sin\theta+\frac{1}{2}-\frac{\cos(2\theta)}{2}\right)d\theta$$
$$= 2+\frac{3\pi}{4}.$$

11.4: #8

$$r = \sin(4\theta)$$
$$A = \int_a^b \frac{1}{2}r^2 d\theta$$
$$= \int_0^{\pi/4}\frac{1}{2}\sin^2(4\theta)d\ theta$$
$$= \frac{1}{2}\left[\theta/2-\frac{1}{16}\sin(8\theta)\right]_0^{\pi/4}$$
$$= \pi/16.$$

11.4: #35

$$r = 1/2+\cos\theta$$

The first step is to find the endpoints for the two regions; that is, solve $r = 0$.

$$\cos\theta = -1/2$$

$$\theta = \pm\frac{2\pi}{3}+2n\pi$$

By examination, we find the larger region is given by $-\frac{2\pi}{3} \le \theta \le \frac{2\pi}{3}$, and the smaller region by $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$.

Inner Region

$$A_{inner} = \int_{2\pi/3}^{4\pi/3}\frac{1}{2}(1/2+\cos\theta)^2 d\theta$$
$$= 1/8\left[3\theta+4\sin\theta+\sin 2\theta\right]_{2\pi/3}^{4\pi/3}$$
$$= 1/8(2\pi-3\sqrt{3})$$

Outer Region

$$A_{outer} = \int_{-2\pi/3}^{2\pi/3}\frac{1}{2}(1/2+\cos\theta)^2 d\theta$$
$$= 1/8\left[3\theta+4\sin\theta+\sin 2\theta\right]_{-2\pi/3}^{2\pi/3}$$
$$= 1/8(4\pi+3\sqrt{3})$$

Thus the area in between is the difference $A = A_{outer} - A_{inner} = 1/8(2\pi+6\sqrt{3})$.

11.4: #46

$$\begin{aligned}
r &= e^{2\theta} \\
\frac{dr}{d\theta} &= 2e^{2\theta} \\
L &= \int_0^{2\pi} \sqrt{r^2 + (\frac{dr}{d\theta})^2 d\theta} \\
&= \int_0^{2\pi} \sqrt{e^{4\theta} + 4e^{4\theta}} d\theta \\
&= \sqrt{5} \int_0^{2\pi} e^{2\theta} d\theta \\
&= \sqrt{5}/2 \left[e^{2\theta} \right]_0^{2\pi} \\
&= \sqrt{5}/2[e^{4\pi} - 1]
\end{aligned}$$

11.4: #55

a. Start with the parametric equation for the surface area of a revolution around the x-axis. Also note that the polar axis ($\theta = 0$) in polar coordinates is the x-axis in Cartesian coordinates.

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Now make the substitutions into polar coordinates: $x = r\cos\theta$ and $y = r\sin\theta$. Note that t is a dummy variable, we could have just called it θ. Also note that:

$$\begin{aligned}
\frac{dy}{d\theta} &= \frac{dr}{d\theta}\sin\theta + r\cos\theta \\
\frac{dx}{d\theta} &= \frac{dr}{d\theta}\cos\theta - r\sin\theta
\end{aligned}$$

Let's look at the quantity $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$.

$$\begin{aligned}
\left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + r^2\cos^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta \\
\left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta + r^2\sin^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta \\
\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 \left(\cos^2\theta + \sin^2\theta\right) + r^2\left(\cos^2\theta + \sin^2\theta\right) \\
\left(\frac{dy}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 &= \left(\frac{dr}{d\theta}\right)^2 + r^2
\end{aligned}$$

Plug in this, and $y = r\sin\theta$ to get

$$S = \int_a^b 2\pi r\sin\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2 d\theta}.$$

b. Now calculate for $r = \sqrt{\cos 2\theta}$.

$$\frac{dr}{d\theta} = \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}$$

$$\begin{aligned}
S &= \int_0^{\pi/4} 2\pi \sqrt{\cos(2\theta)} \sin(\theta) \sqrt{\frac{\sin^2 2\theta}{\cos 2\theta} + \cos 2\theta d\theta} \\
&= \int_0^{\pi/4} 2\pi \sqrt{\cos(2\theta)} \sin(\theta) \sqrt{\frac{\sin^2 2\theta + \cos^2 2\theta}{\cos 2\theta}} d\theta \\
&= \int_0^{\pi/4} 2\pi \sqrt{\cos(2\theta)} \sin(\theta) \sqrt{\frac{1}{\cos 2\theta}} d\theta \\
&= \int_0^{\pi/4} 2\pi \sin(\theta) d\theta \\
S &= -2\pi[\cos(\pi/4) - \cos(0)] = 2\pi(1 - 1/\sqrt{2}).
\end{aligned}$$