MATH 255 **Applied Honors Calculus III** **Winter 2011**

Homework 2

Due: Mon. Jan 24, 2011

Problems
Section 13.1, pg. 833: 6, 12, 14, 18, 22, 40.
Section 13.2, pg. 841: 18, 20, 30, 34, 44.
Section 13.3, pg. 848: 8, 10, 41, 50, 55, 56.
Section 13.4, pg. 856: 2, 6, 36, 42.

Solutions

13.1: #6
a. In $\mathbb{R}^2$, $x = 4$ means that y can have any value. Thus it indicates a vertical line. In $\mathbb{R}^3$, $x = 4$ means that y and z can have any value. Thus it indicates a vertical plane parallel to the yz plane, 4 units above the origin.

b. In $\mathbb{R}^3$, $y = 3$ is a vertical plane parallel to the xz plane, and z=5 is a horizontal plane parallel to xy plane. But $(y = 3, z = 5)$ is the intersection of these two planes, and gives a line, because x can have any value. This line is parallel to the x-axis, passing the point $(0, 3, 5)$.

13.1: #12

$$(x-6)^2 + (y-5)^2 + (z+2)^2 = 7$$

The sphere doesn't intersect the x=0 or y=0 planes (6 and 5 are both larger than $\sqrt{7}$). It does intersect the $z = 0$ plane. Plug in $z = 0$ and get:

$$(x-6)^2 + (y-5)^2 + (2)^2 = 7$$
$$(x-6)^2 + (y-5)^2 = 3$$

The intersection is a circle with radius $= \sqrt{3}$ and center $(6, 5)$ in the xy plane.

13.1: #14

If the sphere passes through the origin, then the radius must be equal to the distance from the sphere center to the origin. E.g. $r^2 = 1^2 + 2^2 + 3^2 = 14$.

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$$

13.1: #18

To find the center and radius of the sphere, we have to complete squares.

$$\begin{aligned}
4x^2 + 4y^2 + 4z^2 - 8x + 16y &= 1 \\
4(x^2 - 2x) + 4(y^2 + 4y) + 4z^2 &= 1 \\
4(x^2 - 2x + 1 - 1) + 4(y^2 + 4y + 4 - 4) + 4z^2 &= 1 \\
4(x^2 - 2x + 1) + 4(y^2 + 4y + 2) + 4z^2 &= 1 + 4 + 16 \\
4(x-1)^2 + 4(y+2)^2 + 4z^2 &= 21 \\
(x-1)^2 + (y+2)^2 + z^2 &= 21/4
\end{aligned}$$

The sphere has radius $\sqrt{21}/2$ and center $= (1, -2, 0)$.

13.1: #22

We need the radius to be equal to the smallest distance from the center of the sphere to each coordinate plane. The smallest distance is 4 (to the $y = 0$ plane), so the radius must be 4.

$$(x-5)^2 + (y-4)^2 + (z-9)^2 = 4^2$$

13.1: #40

We have two distances to consider: from P to A, which we will denote D_{PA} and from P to B, which we denote D_{PB}. $A = (-1, 5, 3), B = (6, 2, -2)$, and we'll give P the coordinates (x, y, z).

$$\begin{aligned}
D_{PA} &= \sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} \\
D_{PB} &= \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2}
\end{aligned}$$

Since $D_{PA} = 2\ D_{PB}$, then $D_{PA}^2 = 4D_{PB}^2$.

$$\begin{aligned}
(x+1)^2 + (y-5)^2 + (z-3)^2 &= 4((x-6)^2 + (y-2)^2 + (z+2)^2) \\
(x+1)^2 + (y-5)^2 + (z-3)^2 - 4((x-6)^2 + (y-2)^2 + (z+2)^2) &= 0
\end{aligned}$$

Expand this out to get (after multiplying by -1)

$$\begin{aligned}
141 - 50x + 3x^2 - 6y + 3y^2 + 22z + 3z^2 &= 0 \\
3(x^2 - 50/3x) + 3(y^2 - 2y) + 3(z^2 + 22/3z) + 141 &= 0 \\
(x^2 - 50/3x) + (y^2 - 2y) + (z^2 + 22/3z) + 47 &= 0
\end{aligned}$$

Complete each square:

$$(x^2 - 50/3x + (25/3)^2 - (25/3)^2 + y^2 - 2y + 1 - 1 + z^2 + 22/3z + (11/3)^2 - (11/3)^2 + 47 = 0$$
$$(x - 25/3)^2 - (25/3)^2 + (y-1)^2 - 1 + (z + 11/3)^2 - (11/3)^2 + 47 = 0$$
$$(x - 25/3)^2 + (y-1)^2 + (z + 11/3)^2 = 332/9$$

Thus we have a circle centered at (25/3, 1, -11/3) with radius $\sqrt{332}/3$.

13.2: #18
a = 2 **i** - 3 **j**
b = **i** + 5 **j**
$|\mathbf{a}| = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$
a + **b** = (2 + 1) **i** + (-3 + 5) **j** = 3 **i** + 2 **j**
a - **b** = (2 - 1) **i** + (-3 - 5) **j** = **i** - 8 **j**
2 **a** = 2 (2 **i** - 3 **j**) = 4 **i** - 6 **j**
3 **a** + 4 **b** = (3 · 2 + 4 · 1) **i** + (3 · (-3) + 4 · 5) **j** = 10 **i** +11 **j**

13.2: #20
$\mathbf{a} = < -3, -4, -1 >$
$\mathbf{b} = < 6, 2, -3 >$
$|\mathbf{a}| = \sqrt{(-3)^2 + (-4)^2 + (-1)^2} = \sqrt{26}$
$\mathbf{a} + \mathbf{b} = < (-3+6), (-4+2), (-1-3) >=< 3, -2, -4 >$
$\mathbf{a}$ - $\mathbf{b} = < (-3-6), (-4-2), (-1+3) >=< -9, -6, 2 >$
$2\ \mathbf{a} = < 2 \cdot (-3), 2 \cdot (-4), 2 \cdot (-1) > = < -6, -8, -2 >$
$3\ \mathbf{a} + 4\ \mathbf{b} = < 3 \cdot (-3) + 4 \cdot 6, 3 \cdot (-4) + 4 \cdot 2, 3 \cdot (-1) + 4 \cdot (-3) >=< 15, -4, -15 >$

13.2: #30
We have the plane's velocity relative to the air, and the air's velocity to the ground. To get the plane's velocity with respect to the ground, we add the two velocity vectors. Once we have the velocity, we can get the speed and direction.

First, find the x-component of the total velocity, by adding the x-components of the given velocities.
$V_x = 250 \cdot \sin(60°) + 50 \cdot \sin(45°) = 250\frac{\sqrt{3}}{2} + 50\frac{\sqrt{2}}{2} = 125\sqrt{3} + 25\sqrt{2} = 251.862$ km/h (Note wind blows "from" $N45°W$)
$V_y = 250 \cdot \cos(60°)$ - $50 \cdot \cos(45°) = 125$ - $\sqrt{2}25 = 89.6$ km/h

The ground speed is the magnitude of the velocity vector:
V = $\sqrt{(125\sqrt{3} + 25\sqrt{2})^2 + (125 - \sqrt{2}25)^2} = 267.340$ km/h.

The true course is found by the direction of the velocity vector:
$\phi = \tan^{-1}(V_x/V_y) = \tan^{-1}(251.862/89.6) = 70.408°$ from the y-axis, which is N70.408°E in the notation of the problem.

13.2: #34
The weight downward is canceled by the total of the vertical components of the tension in the wires. That is W = 2·sin(37)·25N = 30.1N.

13.2: #44

Let's start with the xy-plane, which will be easier to draw. Consider figure 1. The components of **a** drawn are the z- and the xy- components. What is the xy-component? It is the vector **a** projected into the xy-plane. Therefore, it has components $< a_1, a_2, 0 >$. The z- components has vector components $a_z =< 0, 0, a_3 >$ (where we'll consider $a_3 < 0$ here).

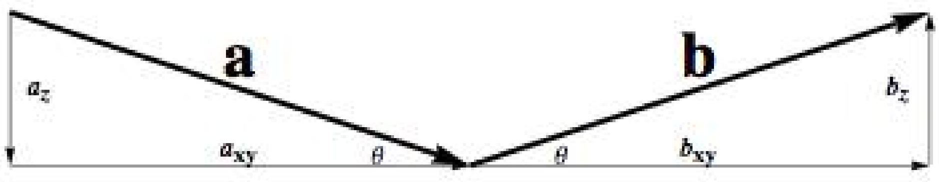

Figure 1: Reflection figure for 13.2 #44.

We are given that the angle of incidence equals the angle of reflection, which we can call θ. What we must intuit is that the length of **a** is equal to the length of **b**. This is true because the light is traveling at a constant speed, so the total distance it travels in a time scan of set duration is always the same.

Therefore, by similar triangles, $| \mathbf{a}_{xy} |=| \mathbf{b}_{xy} |$ and $| \mathbf{a}_z |=| \mathbf{b}_z |$. From our drawing, it is clear that $\mathbf{a}_{xy}$ has the same direction as $\mathbf{b}_{xy}$, but that the direction of the z-components is reversed.

Therefore, a vector incident on the xy-plane mirror changes from $\mathbf{a} =< a_1, a_2, a_3 >$ to $\mathbf{b} =< a_1, a_2, -a_3 >$. By symmetry then, the xz-plane changes the sign of the y-component and the yz-plane changes the sign of the x-component.

In the figure in the book, the light strikes each coordinate plan precisely once, therefore the exit vector $\mathbf{c} =< -a_1, -a_2, -a_3 >= -\mathbf{a}$ and is therefore parallel. (A physicist would use the term anti-parallel, indicating that the line segments of the vectors are parallel, but they point in opposite directions).

13.3: #8

a = 4 **j** - 3 **k**

b = 2 **i** + 4 **j** + 6 **k**

a·**b** = (4 · 4) + (0 · 4) + (-3 · 6) = -2

13.3: #10

|**a**| = 4

|**b**| = 4

angle = 120°.

a·**b** = 4 · 10 · cos(120°) = -20

13.3: #41

We define orth$_{\mathbf{a}}$ $\mathbf{b} = \mathbf{b}$ - proj$_{\mathbf{a}}$ $\mathbf{b}$
proj$_{\mathbf{a}}$ $\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$
To prove that $\mathbf{a}$ is orthogonal to orth$_{\mathbf{a}}$ $\mathbf{b}$, we show $\mathbf{a}$ · orth$_{\mathbf{a}}$ $\mathbf{b} = 0$.

$$\begin{aligned}
\mathbf{a}\cdot(\mathbf{b}-\text{proj}_{\mathbf{a}}\mathbf{b}) &= \mathbf{a}\cdot(\mathbf{b}-\tfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}) \\
&= \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{a}\tfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2} \\
&= \mathbf{a}\cdot\mathbf{b}-\tfrac{\mathbf{a}\cdot\mathbf{a}}{|\mathbf{a}|^2}\mathbf{a}\cdot\mathbf{b} \\
&= \mathbf{a}\cdot\mathbf{b}-\tfrac{|\mathbf{a}|^2}{|\mathbf{a}|^2}\mathbf{a}\cdot\mathbf{b} \\
&= \mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{b} \\
\mathbf{a}\cdot(\mathbf{b}-\text{proj}_{\mathbf{a}}\mathbf{b}) &= 0
\end{aligned}$$

13.3: #50

$$\begin{gathered}
(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0 \\
=< x-a_1, y-a_2, z-a_3 > \cdot < x-b_1, y-b_2, z-b_3 > \\
= (x-a_1)(x-b_1)+(y-a_2)(y-b_2)+(z-a_3)(z-b_3) \\
= x^2-(a_1+b_1)x+a_1b_1+y^2-(a_2+b_2)y+a_2b_2+z^2-(a_3+b_3)z+a_3b_3
\end{gathered}$$

Let's look at the term $x^2-(a_1+b_1)x+a_1b_1$ and complete the square. This will tell us how to do it for the other terms, too.

$$\begin{aligned}
x^2-(a_1+b_1)x+a_1b_1 &= x^2-2\tfrac{(a_1+b_1)}{2}x+a_1b_1/2+a_1b_1/2+a_1^2/4-a_1^2/4+b_1^2/4-b_1^2/4 \\
&= \left(x-\tfrac{(a_1+b_1)}{2}\right)^2+a_1b_1/2-a_1^2/4-b_1^2/4
\end{aligned}$$

Putting this term (and similar terms for y and z) back into the equation above:

$$\begin{gathered}
(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0 \\
= \left(x-\tfrac{(a_1+b_1)}{2}\right)^2+a_1b_1/2-a_1^2/4-b_1^2/4+\left(y-\tfrac{(a_2+b_2)}{2}\right)^2+a_2b_2/2-a_2^2/4-b_2^2/4+ \\
\left(z-\tfrac{(a_3+b_3)}{2}\right)^2+a_3b_3/2-a_3^2/4-b_3^2/4 \\
= \left(x-\tfrac{(a_1+b_1)}{2}\right)^2+\left(y-\tfrac{(a_2+b_2)}{2}\right)^2+\left(z-\tfrac{(a_3+b_3)}{2}\right)^2+a_1b_1/2- \\
a_1^2/4-b_1^2/4+a_2b_2/2-a_2^2/4-b_2^2/4+a_3b_3/2-a_3^2/4-b_3^2/4
\end{gathered}$$

This all equals zero, so we can subtract the constants over to the other side:

$$\begin{gathered}
\left(x-\frac{(a_1+b_1)}{2}\right)^2+\left(y-\frac{(a_2+b_2)}{2}\right)^2+\left(z-\frac{(a_3+b_3)}{2}\right)^2 = \\
-a_1b_1/2+a_1^2/4+b_1^2/4-a_2b_2/2+a_2^2/4+b_2^2/4-a_3b_3/2+a_3^2/4+b_3^2/4
\end{gathered}$$

The right hand side can be written more compactly with dot products:

$$\left(x - \frac{(a_1+b_1)}{2}\right)^2 + \left(y - \frac{(a_2+b_2)}{2}\right)^2 + \left(z - \frac{(a_3+b_3)}{2}\right)^2 = \mathbf{a}\cdot\mathbf{a}/4 + \mathbf{a}\cdot\mathbf{a}/4 - \mathbf{a}\cdot\mathbf{b}/2$$

So, this is a circle with center $(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \frac{a_3+b_3}{2})$ and radius $= \sqrt{\mathbf{a}\cdot\mathbf{a}/4 + \mathbf{b}\cdot\mathbf{b}/4 - \mathbf{a}\cdot\mathbf{b}/2}$.

13.3: #55

2. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$ Use the definition and the commutitive property of multiplication. Use the notation $\mathbf{a} = < a_1, a_2, a_3 >$.

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_1b_1 + a_2b_2 + a_3b_3 \\ &= b_1a_1 + b_2a_2 + b_3a_3 \\ &= \mathbf{b}\cdot\mathbf{a} \end{aligned}$$

4. $(c\mathbf{a}\cdot\mathbf{b}) = c(\mathbf{a}\cdot\mathbf{b}) = \mathbf{a}\cdot(c\mathbf{b})$ Same sort of procedure.

$$\begin{aligned} (c\mathbf{a})\cdot\mathbf{b} &= ca_1b_1 + ca_2b_2 + ca_3b_3 \\ &= c(a_1b_1 + a_2b_2 + a_3b_3) = c(\mathbf{a}\cdot\mathbf{b}) \\ &= a_1cb_1 + a_2cb_2 + a_3cb_3 = \mathbf{a}\cdot(c\mathbf{b}) \end{aligned}$$

5. $\mathbf{0}\cdot\mathbf{a} = 0$. Recall that $\mathbf{0}$ is the vector of all zeros, that is (for 3D) $\mathbf{0} = < 0, 0, 0 >$.

$$\mathbf{0}\cdot\mathbf{a} = 0a_1 + 0a_2 + 0a_3 = 0$$

13.3: #56

What are the diagonals? If **a** is the vector along one edge of the rhombus to the origin, and **b** is the vector along the other edge that meets the origin, then one diagonal is (**a** + **b**) and the other is (**a** - **b**). (See figure 2). To prove they are perpendicular, take the dot product.

$$\begin{aligned} (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{b} \\ &= \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} \\ &= |\,\mathbf{a}\,|^2 - |\,\mathbf{b}\,|^2 \end{aligned}$$

Since this is a rhombus (the sides are of equal length), $|\,\mathbf{a}\,| = |\,\mathbf{b}\,|$, so this equals 0.

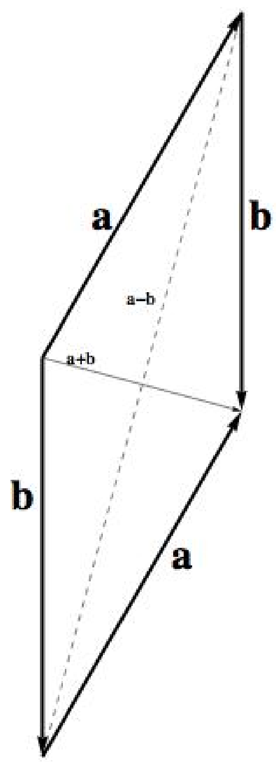

Figure 2: Rhombus for 13.3 #56.

13.4: #2

$\mathbf{a} =< 5, 1, 4 >, \mathbf{b} =< -1, 0, 2 >.$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 4 \\ -1 & -0 & 2 \end{vmatrix} = \mathbf{i}(2-0) - \mathbf{j}(10+4) + \mathbf{k}(0 - -1)$$

$$\mathbf{a} \times \mathbf{b} = 2\mathbf{i} - 14\mathbf{j} + \mathbf{k}$$

13.4: #6

$\mathbf{a} =< 1, e^t, e^{-t} >, \mathbf{b} =< 2, e^t, -e^{-t} >.$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^t & e^{-t} \\ 2 & e^t & -e^{-t} \end{vmatrix} = \mathbf{i}(-1-1) - \mathbf{j}(-e^{-t} - 2e^{-t}) + \mathbf{k}(e^t - 2e^t)$$

$$\mathbf{a} \times \mathbf{b} = -2\mathbf{i} + 3e^{-t}\mathbf{j} - e^{-t}\mathbf{k}$$

13.4: #36

The lever arm $\mathbf{r}$ is the displacement from the pivot to the point where the force is applied.

$$\mathbf{r} = < 4, -4, 0 > .$$

We can now either find the components of the force and compute the cross product, then afterwards find its magnitude. Or, we can use the magnitude of each vector and the angle. We'll take the second approach.

$$\begin{aligned}
|\, \mathbf{r} \,| &= \sqrt{4^2 + 4^2} = \sqrt{32} \\
|\, \mathbf{F} \,| &= 36\ lbs \\
|\, \mathbf{r} \times \mathbf{F} \,| &= |\, \mathbf{r} \,||\, \mathbf{F} \,| \sin(\theta) \\
&= \sqrt{32} \cdot 36 \cdot \sin(45^\circ + 60^\circ) \\
&= 196.708\ ft \cdot lbs
\end{aligned}$$

13.4: #42

Say $\mathbf{a} = < a_1, a_2, a_3 >$, and similarly for $\mathbf{b}$ and $\mathbf{c}$.

Then $\mathbf{b} \times \mathbf{c} = < b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 >$, using the definition on page 850 to speed things along. Thus

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = < & a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), \\
& a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), \\
& a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) >
\end{aligned}$$

Distribute out the terms.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = < & a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 - a_3b_1c_3, \\
& a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 - a_1b_2c_1, \\
& a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 - a_2b_3c_2 >
\end{aligned}$$

Recollect the terms.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = < & b_1(a_2c_2 + a_3c_3) - c_1(a_2b_2 + a_3b_3), \\
& b_2(a_1c_1 + a_3c_3) - c_2(a_1b_1 + a_3b_3), \\
& b_3(a_1c_1 + a_2c_2) - c_3(a_1b_1 + a_2b_2) >
\end{aligned}$$

Add and subtract $a_1b_1c_1$ to the first component, and similarly for the other components.

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = < & b_1(a_1c_1 + a_2c_2 + a_3c_3) - c_1(a_1b_1 + a_2b_2 + a_3b_3), \\
& b_2(a_1c_1 + a_2c_2 + a_3c_3) - c_2(a_1b_1 + a_2b_2 + a_3b_3), \\
& b_3(a_1c_1 + a_2c_2 + a_3c_3) - c_3(a_1b_1 + a_2b_2 + a_3b_3) >
\end{aligned}$$

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = < & b_1(\mathbf{a} \cdot \mathbf{c}) - c_1(\mathbf{a} \cdot \mathbf{b}), \\
& b_2(\mathbf{a} \cdot \mathbf{c}) - c_2(\mathbf{a} \cdot \mathbf{b}), \\
& b_3(\mathbf{a} \cdot \mathbf{c}) - c_3(\mathbf{a} \cdot \mathbf{b}) >
\end{aligned}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$