**MATH 255** **Applied Honors Calculus III** **Winter 2011**

## Homework 4

- Section 14.1, Pages 891–892: #2, #6, #12, and #40.
- Section 14.2, Pages 897–898: #10, #12, #28 (no graphing required), #42, #44, and #47.
- Section 14.3, Pages 904–906, #2, #4, #20, #40, and #50.
- Section 14.4, Pages 914–916: #6, #16, #22, #34, and #40.

### Solutions

**14.1: #2**

For ln(9-$t^2$), we need $9 - t^2 < 0$, so $9 < t^2$. That is, $3 > t$ and $-3 < t$.
For $\frac{t-2}{t+2}$, we need $t + 2 \neq 0$, or $t \neq -2$. So, the domain is: $(-3, -2) \cup (-2, 3)$.

**14.1: #6**

$$\begin{aligned}\lim_{t\to\infty}\langle \tan^{-1}(t), e^{-2t}, \frac{\ln(t)}{t}\rangle &= \langle \lim_{t\to\infty} \tan^{-1}(t), \lim_{t\to\infty} e^{-2t}, \lim_{t\to\infty} \tfrac{\ln(t)}{t}\rangle \\ &= \langle \tfrac{\pi}{2}, 0, \lim_{t\to\infty} \tfrac{1/t}{1}\rangle \\ &= \langle \tfrac{\pi}{2}, 0, 0\rangle\end{aligned}$$

Note the use of L'Hospital's rule in the third component.

**14.1: #12**

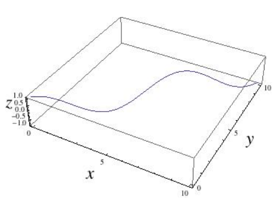


Figure 1: Plot for 14.1 #12.

**14.1: #40**

$$\begin{aligned}\mathbf{r_1}(t_1) &= \langle t_1, t_1^2, t_1^3\rangle \\ \mathbf{r_2}(t_2) &= \langle 1 + 2t_2, 1 + 6t_2, 1 + 14t_2\rangle\end{aligned}$$

We have an intersection when $\mathbf{r_1}(t_1) = \mathbf{r_2}(t_2)$ for some $t_1$ and $t_2$. It's a collision if the equality holds, and $t_1 = t_2$. We set the equations component by component.

$$t_1 = 1 + 2 \cdot t_2$$
$$t_1^2 = 1 + 6t_2$$
$$t_2^3 = 1 + 14t_2$$

Substitute the first eqn into the second to get $1 + 4t_2 + 4t_2^2 = 1 + 6t_2$, and solve for $t_2 = 0$ or $1/2$, which means (from the first eqn) $t_1 = 1$ or $2$. Now, substitute this into the third equation. If you get equality, then these values of $t_2$ are solutions for the vector equations. If not, they aren't.

$$1^3 = 1 + 14 \cdot 0$$
$$1 = 1$$

True.

$$2^3 = 1 + 14 \cdot 1/2$$
$$8 = 8$$

Also true. So there are two intersections (plug in for $\mathbf{r_1}(t_1)$ or $\mathbf{r_2}(t_2)$),
$t_1 = 1$, $t_2 = 0$, $\mathbf{r} = \langle 1, 1, 1 \rangle$
$t_1 = 2$, $t_2 = 1/2$, $\mathbf{r} = \langle 2, 4, 8 \rangle$
but no collisions.

**14.2: #10**

$\mathbf{r}(t) = \langle \cos(3t), t, \sin(3t) \rangle$
$\mathbf{r}'(t) = \langle -3\sin(3t), 1, 3\cos(3t) \rangle$

**14.2: #12**

$\mathbf{r}(t) = \langle \sin^{-1}(t), \sqrt{1-t^2}, 1 \rangle$
$\mathbf{r}'(t) = \langle \frac{1}{\sqrt{1-t^2}}, \frac{-t}{\sqrt{1-t^2}}, 0 \rangle$
See the table in the back of the book for the first term.

**14.2: #28**

$$x = \cos(t)$$
$$y = 3e^{2t}$$
$$z = 3e^{-2t}$$

The point (1,3,3) is found only for $t = 0$ ($y$ and $z$ are both monotone functions of $t$).

$$x' = -\sin(t)$$
$$y' = 6e^{2t}$$
$$z' = -6e^{-2t}$$

Plug in t = 0, get

$$x' = 0$$
$$y' = 6$$
$$z' = -6$$

so, the equations for the tangent line, using $x = x_0 + at$, etc.

$$x_t = 1$$
$$y_t = 3 + 6t$$
$$z_t = 3 - 6t$$

**14.2: #42**

Prove $\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$. Let's say $\mathbf{u}(t) = \langle a(t), b(t), c(t)\rangle$. Then

$$\begin{aligned}
\frac{d}{dt}\left[f(t)\mathbf{u}(t)\right] &= \tfrac{d}{dt}\langle f(t)a(t), f(t)b(t), f(t)c(t)\rangle \\
&= \langle \tfrac{d}{dt}\left[f(t)a(t)\right], \tfrac{d}{dt}\left[f(t)b(t)\right], \tfrac{d}{dt}\left[f(t)c(t)\right]\rangle \\
&= \langle f'(t)a(t) + f(t)a'(t), f'(t)b(t) + f(t)b'(t), f'(t)c(t) + f(t)c'(t)\rangle \\
&= \langle f'(t)a(t), f'(t)b(t), f'(t)c(t)\rangle + \langle f(t)a'(t), f(t)b'(t), f(t)c'(t)\rangle \\
&= f'(t)\langle a(t), b(t), c(t)\rangle + f(t)\langle a'(t), b'(t), c'(t)\rangle \\
&= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)
\end{aligned}$$

**14.2: #44**

Prove $\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$. Say $\mathbf{u}(t) = \langle a(t), b(t), c(t)\rangle$ and $\mathbf{v}(t) = \langle f(t), g(t), h(t)\rangle$. We'll start with the definition of the cross product on page 850.

$$\frac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \frac{d}{dt}\langle b(t)h(t) - c(t)g(t), c(t)f(t) - a(t)h(t), a(t)g(t) - b(t)f(t)\rangle$$

$$\begin{aligned}
&= \langle b(t)'h(t) + b(t)h'(t) - c'(t)g(t) - c(t)g'(t), \\
&\quad c'(t)f(t) + c(t)f'(t) - a'(t)h(t) - a(t)h'(t), \\
&\quad a'(t)g(t) + a(t)g'(t) - b'(t)f(t) - b(t)f'(t)\rangle \\
&= \langle b(t)'h(t) - c'(t)g(t), c'(t)f(t) - a'(t)h(t), a'(t)g(t) - b'(t)f(t)\rangle + \\
&\quad \langle b(t)h'(t) - c(t)g'(t), c(t)f'(t) - a(t)h'(t), a(t)g'(t) - b(t)f'(t)\rangle \\
&= \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)
\end{aligned}$$

**14.2: #44**

$\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}'(t) \times \mathbf{r}'(t) + \mathbf{r}(t) \times \mathbf{r}''(t)$ by Formula 5 of Theorem 3. Since the cross product of vectors in the same direction is 0, the first term disappeared.

**14.3: #2**

$L = \int_a^b \mid \mathbf{r}'(t) \mid dt$. We've got $\mathbf{r}(t) = \langle t^2, \sin(t) - t\cos(t), \cos(t) + t\sin(t)\rangle$, from a = 0 to b = $\pi$.

$$\begin{aligned} \mathbf{r}'(t) &= \langle 2t, \cos(t) - \cos(t) + t\sin(t), -\sin(t) + \sin(t) + t\cos(t)\rangle \\ &= \langle 2t, t\sin(t), t\cos(t)\rangle \end{aligned}$$

$$\begin{aligned} \mid \mathbf{r}'(t) \mid &= \sqrt{(2t)^2 + (t\sin(t))^2 + (t\cos(t))^2} \\ &= \sqrt{4t^2 + t^2} \\ &= t\sqrt{5} \end{aligned}$$

So $L = \int_0^\pi t\sqrt{5}dt = \frac{\pi^2\sqrt{5}}{2}$.

**14.3: #4**

$L = \int_a^b \mid \mathbf{r}'(t) \mid dt$. We've got $\mathbf{r}(t) = \langle t^2, 2t, \ln(t)\rangle$, from a = 1 to b = e.

$$\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$$

$$\begin{aligned} \mid \mathbf{r}'(t) \mid &= \sqrt{(2t)^2 + 2^2 + (1/t)^2} \\ &= \sqrt{4t^2 + 4 + 1/t^2} \\ &= t\sqrt{4 + 4/t^2 + 1/t^4} \\ &= t\sqrt{(1/t^2 + 2)^2} \\ &= t(1/t^2 + 2) \\ &= 1/t + 2t \end{aligned}$$

So $L = \int_0^e (1/t + 2t)dt = \ln(e) + 2e^2/2 - ln(1) - 2 \cdot 1^2/2 = e^2$.

**14.3: #20**

The goal is to find $\kappa$. We have a few equations for that. Which to use? Well, let's calculate a bit first, and maybe it'll be clear which is easiest.

$$\mathbf{r}(t) = \langle e^t\cos(t), e^t\sin(t), t\rangle$$

$$\mathbf{r}'(t) = \langle e^t\cos(t) - e^t\sin(t), e^t\sin(t) + e^t\cos(t), 1\rangle$$

$$\mid \mathbf{r}'(t) \mid = \sqrt{e^{2t}\cos^2(t) + e^{2t}\sin^2(t) - e^{2t}\sin(t)\cos(t) + e^{2t}sin^2(t) + e^{2t}\cos^2(t) + e^{2t}\sin(t)\cos(t) + 1}$$

$$\mid \mathbf{r}'(t) \mid = \sqrt{e^{2t}(\cos^2(t) + \sin^2(t) + \sin^2(t) + \cos^2(t)) + 1}$$

$$\mid \mathbf{r}'(t) \mid = \sqrt{2e^{2t} + 1}$$

I don't think I want to take derivatives like $\frac{d}{dt}\frac{e^t\cos(t)}{\sqrt{2e^{2t}+1}}$, so let's use: $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$.

$$\mathbf{r}'(t) = \langle e^t\cos(t) - e^t\sin(t), e^t\sin(t) + e^t\cos(t), 1\rangle$$
$$\mathbf{r}''(t) = \langle e^t\cos(t) - e^t\sin(t) - e^t\sin(t) - e^t\cos(t), e^t\sin(t) + e^t\cos(t) + e^t\cos(t) - e^t\sin(t), 0\rangle$$
$$\mathbf{r}''(t) = \langle -2e^t\sin(t), 2e^t\cos(t), 0\rangle$$

Now calculate the cross product.

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ e^t(\sin(t) - \cos(t) & e^t(\sin(t) + \cos(t)) & 1 \\ -2e^t\sin(t) & 2e^t\cos(t) & 0 \end{vmatrix}$$

$$= \mathbf{i}(0+2e^t\cos(t)) - \mathbf{j}(-2e^t\sin(t)) + \mathbf{k}(e^{2t}(2\cos^2(t) - 2\cos(t)\sin(t) + 2\sin^2(t) + 2\cos(t)\sin(t)) = 2\langle e^t\cos(t), e^t\sin(t$$

The magnitude $|\ \mathbf{r}'(t)\times\mathbf{r}''(t)\ | = 2\sqrt{e^{2t}(\cos^2(t)+\sin^2(t)) + e^{4t}} = 2e^t\sqrt{1+e^{2t}}$.
So, $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2e^t\sqrt{1+e^{2t}}}{(2e^{2t}+1)^{3/2}}$
At (x,y,z) = (1,0,0), t = 0. $\kappa(0) = \frac{2\sqrt{1+1}}{(2\cdot 1+1)^{3/2}} = (\frac{2}{3})^{3/2}$.

**14.3: #40**

$\mathbf{r}(t) = \langle e^t, e^t\sin(t), e^t\cos(t)\rangle$. At the point (1,0,1) t = 0.

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{|\ \mathbf{r}'(t)\ |}$$
$$\mathbf{r}'(t) = \langle e^t, e^t\left[\sin(t)+\cos(t)\right], e^t\left[\cos(t)-\sin(t)\right]\rangle$$
$$|\ \mathbf{r}'(t)\ | = \sqrt{e^{2t} + e^{2t}(\sin^2(t) + \cos^2(t) + 2\sin(t)\cos(t) + \sin^2(t) + \cos^2(t) - 2\sin(t)\cos(t))}$$
$$|\ \mathbf{r}'(t)\ | = \sqrt{e^{2t} + 2e^{2t}}$$
$$|\ \mathbf{r}'(t)\ | = e^t\sqrt{3}$$

$$\mathbf{T} = \frac{\langle e^t, e^t\left[\sin(t)+\cos(t)\right], e^t\left[\cos(t)-\sin(t)\right]\rangle}{e^t\sqrt{3}}$$
$$\mathbf{T} = \frac{1}{\sqrt{3}}\langle 1, \sin(t)+\cos(t), \cos(t)-\sin(t)\rangle$$

$$\mathbf{T}' = \frac{1}{\sqrt{3}}\langle 0, \cos(t) - \sin(t), -\sin(t) - \cos(t)\rangle$$

$$\mid \mathbf{T}' \mid = \frac{1}{\sqrt{3}}\sqrt{\cos^2(t) + \sin^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t)}$$

$$\mid \mathbf{T}' \mid = \sqrt{\frac{2}{3}}$$

$$\mathbf{N} = \frac{\mathbf{T}'}{\mid \mathbf{T}' \mid}$$

$$\mathbf{N} = \frac{1}{\sqrt{2}}\langle 0, \cos(t) - \sin(t), -\sin(t) - \cos(t)\rangle$$

$\mathbf{B} = \mathbf{T} \times \mathbf{N}$, so

$$\mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{6}}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & \sin(t) + \cos(t) & \cos(t) - \sin(t) \\ 0 & \cos(t) - \sin(t) & -\sin(t) - \cos(t) \end{vmatrix}$$

$$\begin{aligned} \mathbf{B}\sqrt{6} = \mathbf{i}\left[-(\sin^2(t) + \cos^2(t) + 2\sin(t)\cos(t)) - (\cos^2(t) + \sin^2(t) - 2\sin(t)\cos(t))\right] \\ -\mathbf{j}(-\sin(t) - \cos(t)) + \mathbf{k}(\cos(t) - \sin(t)) \\ = \langle -2, \sin(t) + \cos(t), \cos(t) - \sin(t)\rangle \end{aligned}$$

$$\mathbf{B} = \frac{1}{\sqrt{6}}\langle -2, \sin(t) + \cos(t), \cos(t) - \sin(t)\rangle$$

At t = 0,

$$\mathbf{T} = \frac{1}{\sqrt{3}}\langle 1, 1, 1\rangle$$

$$\mathbf{N} = \frac{1}{\sqrt{2}}\langle 0, 1, -1\rangle$$

$$\mathbf{B} = \frac{1}{\sqrt{6}}\langle -2, 1, 1\rangle$$

**14.3: #50**

$$
\begin{aligned}
\mathbf{N} &= \mathbf{B}\times\mathbf{T}\\
\frac{d}{ds}\mathbf{N} &= \tfrac{d}{ds}\left[\mathbf{B}\times\mathbf{T}\right]\\
\frac{d\mathbf{N}}{ds} &= \tfrac{d\mathbf{B}}{ds}\times\mathbf{T}+\mathbf{B}\times\tfrac{d\mathbf{T}}{ds}\\
&= -\tau\mathbf{N}\times\mathbf{T}+\mathbf{B}\times\kappa\mathbf{N}\\
&= \tau\mathbf{T}\times\mathbf{N}+\kappa\mathbf{B}\times\mathbf{N}\\
&= \tau\mathbf{B}+\kappa\mathbf{B}\times\mathbf{N}
\end{aligned}
$$

Now, the question is, what is $\mathbf{B}\times\mathbf{N}$? There are two approaches: substituting $\mathbf{N}=\mathbf{B}\times\mathbf{T}$, so $\mathbf{B}\times\mathbf{N}=\mathbf{B}\times(\mathbf{B}\times\mathbf{T})=(\mathbf{B}\cdot\mathbf{T})\mathbf{B}-(\mathbf{B}\cdot\mathbf{B})\mathbf{T}$. $\mathbf{B}$ is orthogonal to $\mathbf{T}$, and has magnitude 1, so $\mathbf{B}\times\mathbf{N}=-\mathbf{T}$. The other way is reasoning through the right hand rule. If $\mathbf{T}\times\mathbf{N}=\mathbf{B}$, then $\mathbf{B}\times\mathbf{T}=\mathbf{N}$ and $\mathbf{N}\times\mathbf{B}=\mathbf{T}$. Either way, $\frac{d\mathbf{N}}{ds}=\tau\mathbf{B}-\kappa\mathbf{T}$.

**14.4: #6**

Given $\mathbf{r}(t)=<\sin t, 2\cos t, 0>$, the velocity is $\mathbf{v}(t)=\mathbf{r}'(t)=<\cos t,-2\sin t,0>$, the speed is $v(t)=|\mathbf{v}(t)|=\sqrt{\cos^2 t+4\sin^2 t}$ and the acceleration is $\mathbf{a}(t)=\mathbf{r}''(t)=<-\sin t,-2\cos t,0>$. Plugging in $t=\pi/6$, we have $\mathbf{v}(\pi/6)=<\sqrt{3}/2,-1,0>$, $v(\pi/6)=\sqrt{7}/2$ and

$$\mathbf{a}(\pi/6)=<-1/2,-\sqrt{3},0>$$

**14.4: #16**

To find the velocity, we write

$$\mathbf{v}(t)=\mathbf{v}(0)+\int_0^t\mathbf{a}(u)du=<1,1,-1>+\int_0^t<0,0,-10>du=<1,1,-1-10t>$$

Similarly, position vector of the particle is

$$\mathbf{r}(t)=\mathbf{r}(0)+\int_0^t\mathbf{v}(u)du=<2,3,0>+\int_0^t<1,1,-1-10u>du=<2+t,3+t,-t-5t^2>$$

See figure .

**14.4: #22**

If a particle moves with constant speed $v(t)=C$, then the velocity satisfies $\mathbf{v}(t)\cdot\mathbf{v}(t)=v^2(t)=C^2$. Now if we differentiate the dot product, we have $\mathbf{v}(t)\cdot\mathbf{v}'(t)+\mathbf{v}'(t)\cdot\mathbf{v}(t)=(C^2)'=0$, and therefore $2\mathbf{v}(t)\cdot\mathbf{v}'(t)=0$. Finally recall that the dot product of two vectors is 0 means that the vectors are orthogonal. We have $\mathbf{v}(t)\cdot\mathbf{v}'(t)=0$ and therefore $\mathbf{v}(t)$ is orthogonal to $\mathbf{v}'(t)$.

**14.4: #34**

We use formulas

$$\mathbf{a}_T=\text{comp}_{\mathbf{v}}\mathbf{a}=\frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|}=\frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

and

$$\mathbf{a}_N=\kappa|\mathbf{v}|^2=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}|\mathbf{r}'(t)|^2=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$

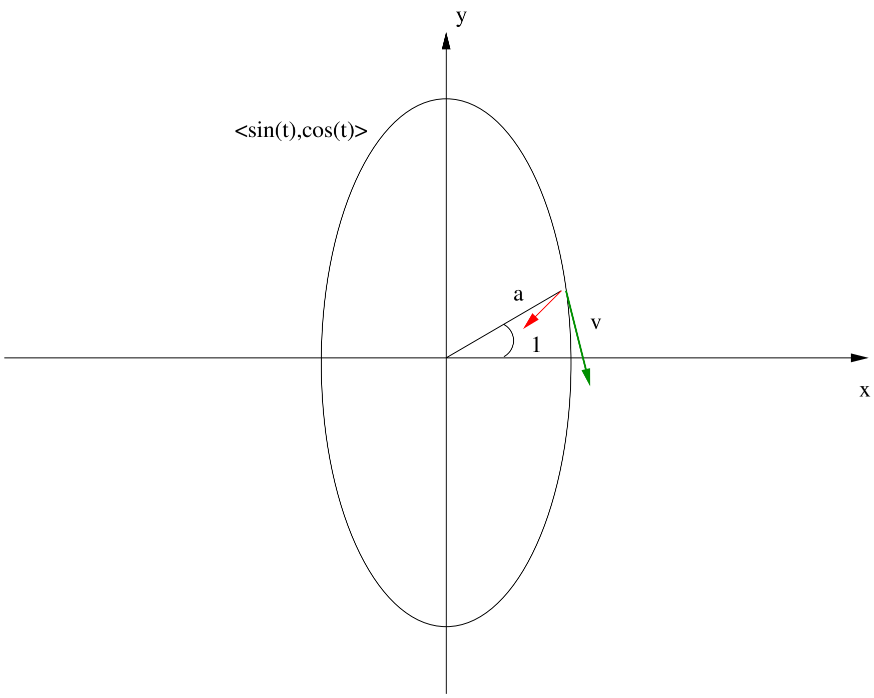


Figure 2:

Since $\mathbf{r}'(t) =< 1, 2t, 3 >, \mathbf{r}''(t) =< 0, 2, 0 >$, we compute $|\mathbf{r}'(t)| = \sqrt{10 + 4t^2}, \mathbf{r}'(t) \cdot \mathbf{r}''(t) = 4t$ and $\mathbf{r}'(t) \times \mathbf{r}''(t) =< -6, 0, 2 >$. Also $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36 + 4} = 2\sqrt{10}$. Finally

$$\mathbf{a}_T = \frac{4t}{\sqrt{10 + 4t^2}}$$

and

$$\mathbf{a}_N = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}$$

**14.4: #40**

(a) We have

$$\frac{d\mathbf{v}}{dt} = \frac{1}{m}\frac{dm}{dt}\mathbf{v}_e$$

To solve for $\mathbf{v}$ we integrate both sides of the equation

$$\mathbf{v}(t) - \mathbf{v}(0) = \int_0^t \frac{1}{m}\frac{dm}{du}\mathbf{v}_e du$$

The right hand side is

$$\int_0^t \frac{1}{m}\frac{dm}{du}\mathbf{v}_e du = \mathbf{v}_e \int_0^t \frac{m'(u)}{m(u)} du = \mathbf{v}_e \ln \frac{m(t)}{m(0)}$$

by substitution. Therefore,

$$\mathbf{v}(t) = \mathbf{v}(0) + \mathbf{v}_e \ln \frac{m(t)}{m(0)}$$

(b) We have $|\mathbf{v}(t)| = 2|\mathbf{v}_e|$ and $|\mathbf{v}(0)| = 0$. By part (a),

$$\mathbf{v}(t) = \mathbf{v}_e \ln \frac{m(t)}{m(0)}$$

and therefore

$$2 = \frac{|\mathbf{v}(t)|}{|\mathbf{v}_e|} = \left| \ln \frac{m(t)}{m(0)} \right|$$

Mass is decreasing, so

$$\left| \ln \frac{m(t)}{m(0)} \right| = \ln \frac{m(0)}{m(t)}$$

and finally

$$m(t) = e^{-2} m(0)$$

Thus $\frac{m(0) - e^{-2} m(0)}{m(0)}$ is the fraction of the initial mass that is burned as fuel.