Homework 6 Solutions

Note: In what follows, numbers in parentheses indicate the problem numbers for users of the sixth edition. A * indicates that this problem is not in the sixth edition and that you should look in the Michigan edition to find it.

Section 15.5, pg. 974: 8 (*), 22, 40, 51 (53).
Section 15.6, pg. 987: 6, 12, 22, 34 (*), 60.
Section 15.7, pg. 997: 28 (30), 36 (38), 40 (*), 52 (54), 54 (56).
Section 15.8, pg. 1007: 10, 30 (*), 40 (42), 43 (45).

Solutions
15.5 #8

$$\begin{aligned} z &= x/y \\ x &= se^t \\ y &= 1 + se^{-t} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= 1/y \cdot se^t + -x/y^2 \cdot (-se^{-t}) \\ &= (1 + se^{-t})^{-1} se^t + s^2(1 + se^{-t})^{-2} \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= 1/y \cdot e^t + -x/y^2 \cdot (e^{-t}) \\ &= (1 + se^{-t})^{-1} e^t - s(1 + se^{-t})^{-2} \end{aligned}$$

15.5 #22

There are three partial derivatives this time, but we can analyze them at the indicated points.

$$\begin{aligned} u &= \sqrt{r^2+s^2} \\ r &= y + x\cos t \\ r(1,2,0) &= 3 \\ s &= x + y\sin t \\ s(1,2,0) &= 1 \end{aligned}$$

$$\begin{array}{rr} r_x = \cos t & s_x = 1 \\ r_x(1,2,0) = 1 & s_x(1,2,0) = 1 \\ r_y = 1 & s_y = \sin t \\ r_y(1,2,0) = 1 & s_y(1,2,0) = 0 \\ r_t = -x\sin t & s_t = y\cos t \\ r_t(1,2,0) = 0 & s_t(1,2,0) = 2 \end{array}$$

$$\begin{array}{rr} u_r = r(r^2+s^2)^{-1/2} & u_s = s(r^2+s^2)^{-1/2} \\ u_r(3,1) = 3/\sqrt{10} & u_s(3,1) = 1/\sqrt{10} \end{array}$$

Finally, we put the piece together

$$\begin{aligned} u_x &= u_r r_x + u_x s_x = 3/\sqrt{10}\cdot 1 + 1/\sqrt{10}\cdot 1 = 4/\sqrt{10} \\ u_y &= u_r r_y + u_s s_y = 3/\sqrt{10}\cdot 1 + 1/\sqrt{10}\cdot 0 = 3/\sqrt{10} \\ u_t &= u_r r_t + u_s s_t = 3/\sqrt{10}\cdot 0 + 1/\sqrt{10}\cdot 2 = 2/\sqrt{10} \end{aligned}$$

15.5 #40

We have $V = IR$ so that

$$\frac{dV}{dt} = \frac{\partial V}{\partial I}\frac{dI}{dt} + \frac{\partial V}{\partial R}\frac{dR}{dt}.$$

Then $\frac{\partial V}{\partial I} = R$ and $\frac{\partial V}{\partial R} = I$ so

$$-.01 = 400\cdot\frac{dI}{dt} + .08\cdot .03$$

from which we calculate that $\frac{dI}{dt} = -3.1\cdot 10^{-5} A/s$.

15.5 #51

Prove

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r}\frac{\partial f}{\partial r}$$

The trick here is to very carefully execute the partial derivatives. It's easier to start with the left side and wind up with the right side. Let's first compute $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial x}$. The partials with respect to r and θ we'll leave, since these are the types of terms we want. We know $x = r\cos\theta$ and $y = r\sin\theta$, and from this we find $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1} y/x$.

$$r = \sqrt{x^2+y^2}$$
$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r} = \cos\theta$$
$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r} = \sin\theta$$

$$\theta = \tan^{-1} y/x$$
$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r}$$
$$\frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\frac{1}{x} = \frac{1}{1+(y/x)^2}\frac{x}{x^2} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

So, the first partial derivatives are $f_x = f_r\cos\theta + f_\theta\frac{-\sin\theta}{r}$ and $f_x = f_r\sin\theta + f_\theta\frac{\cos\theta}{r}$
When we compute the second derivatives, we have to do the chain rule *again.*
$f_{xx} = \frac{\partial}{\partial x}f_x = \frac{\partial f_x}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f_x}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial f_x}{\partial r}\cos\theta + \frac{\partial f_x}{\partial \theta}\frac{-\sin\theta}{r}$
We plug in for

$\frac{\partial f_x}{\partial r} = \frac{\partial}{\partial r}(f_r\cos\theta + f_\theta\frac{-\sin\theta}{r}) = f_{rr}\cos\theta - f_{\theta r}\frac{\sin\theta}{r} + f_\theta\frac{\sin\theta}{r^2}$
and $\frac{\partial f_x}{\partial \theta} = \frac{\partial}{\partial \theta}(f_r\cos\theta - f_\theta\frac{\sin\theta}{r}) = f_{\theta r}\cos\theta - f_r\sin\theta - f_{\theta\theta}\frac{\sin\theta}{r} - f_\theta\frac{\cos\theta}{r}$.
Thus we get $f_{xx} = f_{rr}\cos^2\theta - 2f_{r\theta}\frac{\sin\theta\cos\theta}{r} + 2f_{\theta\theta}\frac{\sin\theta\cos\theta}{r^2} + f_r\frac{\sin^2\theta}{r} + f_{\theta\theta}\frac{\sin^2\theta}{r^2}$.
Do the same process to get $f_{yy} = f_{rr}\sin^2\theta + 2f_{r\theta}\frac{\sin\theta\cos\theta}{r} - 2f_{\theta\theta}\frac{\sin\theta\cos\theta}{r^2} + f_r\frac{\cos^2\theta}{r} + f_{\theta\theta}\frac{\cos^2\theta}{r^2}$ and then find the sum:

$$f_{xx} + f_{yy} = (\cos^2\theta + \sin^2\theta)(f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2}) = f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2}$$

15.6 #6 $\nabla f(2,0) = (\sin(xy) + x\cos(xy)y, x\cos(xy)x)(2,0) = (0,4)$. The unit vector in the direction of θ is $(1/2, \sqrt{3}/2)$ so that the directional derivative in the direction of θ is $2\sqrt{3}$.

15.6 #12 $\nabla g(2,1) = ((x^2+y^2)^{-1}(2x), (x^2+y^2)^{-1}(2y))(2,1) = (4/5, 2/5)$. The unit vector in the direction of $\vec{v}$ is $(-1/\sqrt{5}, 2/\sqrt{5})$ so that the directional derivative in the direction of $\vec{v}$ is 0.

15.6 #22 The maximum rate of change occurs in the direction of the gradient. We have $\nabla(f)(0,0) = (-qe^{-p} + e^{-q}, e^{-p} - pe^{-1})(0,0) = (1,1)$. Therefore the maximum rate of change occurs in the direction $(1,1)$ and that magnitude is the directional derivative in that direction, which is $(1,1)\cdot(1/\sqrt{2}, 1/\sqrt{2}) = 2/\sqrt{2}$.

15.6 #34 We will first need to know $\nabla z(50,80) = (-.02x, -.04y)(50,80) = (-1, -3.2)$.
In part a) we are asked to compute the directional derivative in the direction $(0,-1)$. This is 3.2 so we will start to ascend at $3.2m/s$. In part b) we are asked to compute the directional derivative in the direction $(-1,1)$. This is $-2.2/\sqrt{2}$ so we will start to descend that $-2.2/\sqrt{2}m/s$.

In part c) we know that the gradient points in the steepest direction, so $(-1, -3.2)$ is the direction of largest slope. The rate of ascent is $(1 + 3.2^2)^{1/2} m/s$. The angle is $\tan^{-1}(1 + 3.2^2)^{1/2}$ (about 1.28 radians) which is pretty steep!

15.6 #60 To find parametric equations for the tangent line, we first have to parameterize the ellipse of intersection. This can be done by

$$r(t) = (\sqrt{5}\cos(t), \sqrt{5}\sin(t), 3 - \sqrt{5}\sin(t)).$$

The point $(1, 2, 1)$ is given by $t = \cos^{-1}(1/\sqrt{5})$. The tangent vector is given by

$$r'(t) = (-\sqrt{5}\sin(t), \sqrt{5}\cos(t), -\sqrt{5}\cos(t)).$$

So the tangent line may be parameterized by

$$L(s) = (1, 2, 1) + s(-2, 1, -1).$$

Graphs omitted.

15.7 #28

The critical points of f are found by setting $f_x = y-1$ and $f_y = x-2$ to zero. The critical point of f is then $(2, 1)$ which is contained in our region. Notice that f_{xx} and f_{yy} are both 0 and $f_{xy} = 1$ so that this point is neither a local max nor a local min. Now we must test the boundary of D. The first boundary is given by $y = 0$ for $1 \leq x \leq 5$. When $y = 0$, $f(x, 0) = 3-x$ which has a max at $f(1, 0) = 2$ and a min at $f(5, 0) = -2$. The second boundary is given by $x = 1$ for $0 \leq y \leq 4$. When $x = 1$, $f(1, y) = 2 - y$ which has a max at $f(1, 0) = 2$ and a min at $f(1, 4) = -2$. The third boundary is given by the line $y = -x + 5$ for $1 \leq x \leq 5$. Along this line our function becomes $f(x, 5-x) = 3 + x(5-x) - x - 2(5-x) = 3 + 5x - x^2 - x - 10 + 2x = -7 + 6x - x^2$. This function has it's minimum at $x = 3$ where the function has value 2. We still have to test the endpoints, which we've already done. So our function has a maximum value of 2 and minimum value of -2 on the region D.

15.7 #36 First compute the critical points: $f_x = 3e^y - 3x^2$ and $f_y = 3xe^y - 3e^{3y}$. Setting these to 0 and solving gives us that the only critical point is $(1, 0)$. To check that it is a local maximum we'll use the second derivative test. $f_{xx} = -6x$, $f_{yy} = 3xe^y - 9e^{3y}$ and $f_{xy} = 3e^y$. At the point $(1, 0)$, calculate that $D = (-6)(-6) - (9)$ from which we conclude that our critical point is a local max ($D > 0$ and $f_{xx} < 0$). But if you hold y constant and let x approach $-\infty$ you will see that the function gets arbitrarily large. Still there are no local minimums!

15.7 #40 We want to minimize the function $D = \sqrt{x^2 + y^2 + z^2}$ subject to the constraint that $x^2y^2z = 1$. As usual, it's enough to minimize the function $D = x^2 + y^2 + z^2$. (This is a standard trick. You just need to observe that $\sqrt{a} < \sqrt{b}$ if and only if $a < b$.) To incorporate the constraint, we substitute $z = 1/(x^2y^2)$ into D. So we want to minimize $D = x^2 + y^2 + (xy)^{-2}$. To do this we compute $D_x = 2x - 2x^{-3}y^{-2}$ and $D_y = 2y - 2y^{-3}x^{-2}$. Setting these both to 0 you get $x^4 = 1/y^2$ and $y^4 = 1/x^2$. Substituting, we see that $x^8 = x^2$. Since x can't be 0, the only solutions are 1 and -1. The same goes for y. Then we have to test and make sure we've found local minima. $D_{xx} = 2 + 6x^{-4}y^{-2}$, $D_{yy} = 2 + 6x^{-2}y^{-4}$ and $D_{xy} = 4x^{-3}y^{-3}$. By plugging in our point, the second derivative tells us that the points $(1, 1, 1)$, $(1, -1, 1)$, $(-1, 1, 1)$ and $(-1, -1, 1)$ are local minima.

15.7 #52 We want to find the maximum value of $P = 2pq + 2pr + 2rq$. Since we know that $r = 1 - p - q$, we must maximize the function of two variables $P = 2pq + (2p + 2q)(1 - p - q) = 2p + 2q - 2p^2 - 2q^2 - 2pq$.

Take the derivatives to find $P_p = 2 - 4p - 2q$ and $P_q = 2 - 4q - 2p$. If you set these each to 0 and solve, you find that $p = q$ so that $p = q = 1/3$ so that $r = 1/3$. By the second derivative test $P_{pp} = -4$, $P_{qq} = -4$ and $P_{pq} = -2$ so that this is a local maximum. We are not finished though, because the variables p and q are constrained (they are percentages, so they can't be arbitrary). The constraint we have is that p and q had better both be positive and less than or equal to 1. So our boundary is given by $p = 0$, $p = 1$, $q = 0$, $q = 1$. When $p = 0$ we have $P(0, q) = 2q - 2q^2 = 2q(1 - q)$. This has a maximum value of 1/2 which is not a maximum. The other boundaries are treated in exactly the same way, but you have to look at each of them and show that they don't contribute a maximum value larger than 2/3.

15.7: #54 Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1, 2, 3)$. If we write the equation of the plane as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the plane intersects $x-$, $y-$ and $z-$ axes at $x = a, y = b$ and $z = c$. Therefore the volume of the tetrahedron is $V = \frac{abc}{6}$. Since $(1, 2, 3)$ must be on the plane, we need $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1(*)$. Consider c as a function of a and b. Then $V_a = \frac{b}{6}(c + a\frac{\delta c}{\delta a})$ and $V_b = \frac{a}{6}(c + b\frac{\delta c}{\delta b})$. Differentiating $(*)$ with respect to a we get $-a^{-2} - 3c^{-2}\frac{\delta c}{\delta a} = 0$. Hence $\frac{\delta c}{\delta a} = \frac{-c^2}{3a^2}$. Similarly, we find that $\frac{\delta c}{\delta b} = \frac{-2c^2}{3b^2}$. Then $V_a = \frac{b}{6}(c + a\frac{-c^2}{3a^2}) = 0$ implies that $c = 3a$ and $V_b = \frac{a}{6}(c + b\frac{-2c^2}{3b^2}) = 0$ implies that $c = 3/2b$. Putting these in $(*)$, we have $a = 3$, $b = 6$ and $c = 9$. Thus the equation is $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$.