

Homework 8

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16.4: #14

$$\begin{aligned}
 \iint_R ye^x dA &= \int_0^5 \int_0^{\pi/2} r \sin(\theta) e^{r \cos(\theta)} r d\theta dr \\
 &= \int_0^5 -re^{r \cos(\theta)} \Big|_0^{\pi/2} dr \\
 &= \int_0^5 -r(1 - e^r) dr \\
 &= \int_0^5 re^r - r dr \\
 &= re^r - e^r - \frac{r^2}{2} \Big|_0^5 \\
 &= 4e^5 - \frac{23}{5}
 \end{aligned}$$

In the fifth line we integrated by parts, but didn't write down all the steps.

16.4: #18 If you draw the picture, you see that you need to take all theta values between 0 and 2π . So the area is given by the formula:

$$\begin{aligned}
 \int_0^{2\pi} \int_0^{4+3\cos(\theta)} r dr d\theta &= \int_0^{2\pi} \frac{1}{2} (4 + 3\cos(\theta))^2 d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} 16 + 24\cos(\theta) + 9\cos^2(\theta) d\theta \\
 &= \frac{1}{2} \int_0^{2\pi} 16 + 24\cos(\theta) + 9\frac{1 + \cos(2\theta)}{2} d\theta \\
 &= \frac{1}{2} (16\theta + 24\sin(\theta) + 9\frac{\theta + \sin(2\theta)/2}{2}) \Big|_0^{2\pi} \\
 &= \frac{1}{2} (32\pi + 9\pi) \\
 &= \frac{41\pi}{2}
 \end{aligned}$$

16.4: #26 The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$, namely, when $x^2 + y^2 = 1$. So the volume we're trying to compute is given by

$$\iint_D (4 - x^2 - y^2) - (3x^2 + 3y^2) dA$$

where D is the region in the plane $x^2 + y^2 \leq 1$. Switching to polar coordinates,

$$\begin{aligned} \int_0^{2\pi} \int_0^1 (4 - 4r^2)r dr d\theta &= \int_0^{2\pi} 2r^2 - r^4 \Big|_0^1 \\ &= \int_0^{2\pi} 1 d\theta \\ &= 2\pi \end{aligned}$$

16.4: #36

a) The integral over the disc of radius a is given by

$$\begin{aligned} \int \int_{D_a} e^{-x^2-y^2} dA &= \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta \\ &= \int_0^{2\pi} \frac{-1}{2} (e^{-a^2} - 1) d\theta \\ &= \pi(1 - e^{-a^2}) \end{aligned}$$

So taking the limit as a approaches infinity gives that this improper integral has value π .

b) The integral over the square S_a may be written

$$\int_{-a}^a \int_{-a}^a e^{-x^2-y^2} dx dy = \int_{-a}^a e^{-x^2} dx \int_{-a}^a e^{-y^2} dy$$

When we take the limit as a approaches infinity, we use the fact that the limit of the product is the product of the limits (as long as the limits are not infinite, which they are not because the functions are bounded). Then from part a) we conclude that

$$\pi = \lim \int_{-a}^a e^{-x^2} dx \int_{-a}^a e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy.$$

c) From part b) we have

$$\pi = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy = \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-x^2} dx.$$

Now take square roots of both sides.

d) If $t = \sqrt{2}x$, then $dt = \sqrt{2}dx$ and so

$$\pi = \int_{-\infty}^{\infty} e^{-t^2/2} dt = \int_{-\infty}^{\infty} e^{-x^2} \sqrt{2} dx = \sqrt{2}\pi.$$

16.5: #7

$$m = \int_0^1 \int_0^{e^x} y dy dx = \frac{1}{2} \int_0^1 y^2 \Big|_{y=0}^{e^x} dx = \frac{1}{2} \int_0^1 e^{2x} dx = \frac{1}{4} e^{2x} \Big|_{x=0}^1 = \frac{e^2 - 1}{4}.$$

$$\bar{x} = \frac{1}{m} \int_0^1 \int_0^{e^x} xy \, dy \, dx = \frac{1}{2m} \int_0^1 xy^2 \Big|_{y=0}^{e^x} = \frac{1}{2m} \int_0^1 xe^{2x} \, dx = \frac{1}{2m} \frac{(2x-1)e^{2x}}{4} \Big|_{x=0}^1 = \frac{1+e^2}{8m} = \frac{e^2+1}{2(e^2-1)}.$$

In computing this integral, we have to integrate by parts, but we skipped the details.

$$\bar{y} = \frac{1}{m} \int_0^1 \int_0^{e^x} y^2 \, dy \, dx = \frac{1}{3m} \int_0^1 y^3 \Big|_{y=0}^{e^x} \, dx = \frac{1}{3m} \int_0^1 e^{3x} \, dx = \frac{1}{9m} e^{3x} \Big|_{x=0}^1 = \frac{1}{9m} (e^3-1) = \frac{4(e^3-1)}{9(e^2-1)}.$$

The center of mass is thus

$$(\bar{x}, \bar{y}) = \left(\frac{e^2+1}{2(e^2-1)}, \frac{4(e^3-1)}{9(e^2-1)} \right).$$

16.5: #12 Here the density function is $\rho(x, y) = k(x^2 + y^2)$ for some constant k . Thus

$$m = k \iint_D (x^2 + y^2) \, dA = k \int_0^{\pi/2} \int_0^1 r^3 \, dr \, d\theta = \frac{k}{4} \int_0^{\pi/2} r^4 \Big|_{r=0}^1 \, d\theta = \frac{k\pi}{8}.$$

$$\bar{x} = \frac{k}{m} \iint_D x(x^2+y^2) \, dA = \frac{k}{m} \int_0^{\pi/2} \int_0^1 r^4 \cos \theta \, dr \, d\theta = \frac{k}{5m} \int_0^{\pi/2} r^5 \cos \theta \Big|_{r=0}^1 \, d\theta = -\frac{k}{5m} \sin \theta \Big|_{\theta=0}^{\pi/2} = \frac{k}{5m} = \frac{8}{5\pi}.$$

$$\bar{y} = \frac{k}{m} \iint_D y(x^2+y^2) \, dA = \frac{k}{m} \int_0^{\pi/2} \int_0^1 r^4 \sin \theta \, dr \, d\theta = \frac{k}{5m} \int_0^{\pi/2} r^5 \sin \theta \Big|_{r=0}^1 \, d\theta = \frac{k}{5m} \cos \theta \Big|_{\theta=0}^{\pi/2} = \frac{k}{5m} = \frac{8}{5\pi}.$$

Thus the center of mass is

$$(\bar{x}, \bar{y}) = \left(\frac{8}{5\pi}, \frac{8}{5\pi} \right).$$

16.5: #24 (a) Clearly $f(x, y)$ is never negative. We just have to show that the integral of this function over \mathbb{R}^2 is 1. In fact, since the function is zero outside of a box, we just have to show that the integral of this function over $[0, 1] \times [0, 1]$ is 1.

$$\int_0^1 \int_0^1 4xy \, dx \, dy = \int_0^1 2x^2y \Big|_{x=0}^1 \, dy = \int_0^1 2y \Big|_{x=0}^1 \, dy = y^2 \Big|_{y=0}^1 = 1.$$

(b) In part (i), we have to integrate f over the region in which $x \geq \frac{1}{2}$. This is

$$\int_0^1 \int_{1/2}^1 4xy \, dx \, dy = \int_0^1 2x^2y \Big|_{x=1/2}^1 \, dy = \int_0^1 2y(1-1/4) \, dy = \frac{3}{4}y^2 \Big|_{y=0}^1 = \frac{3}{4}.$$

In part (ii), we have to integrate f over the region in which $x \geq \frac{1}{2}$ and $y \leq \frac{1}{2}$. This is

$$\int_0^{1/2} \int_{1/2}^1 4xy \, dx \, dy = \int_0^{1/2} 2x^2y \Big|_{x=1/2}^1 \, dy = \int_0^{1/2} 2y(1-1/4) \, dy = \frac{3}{4}y^2 \Big|_{y=0}^{1/2} = \left(\frac{3}{4}\right) \left(\frac{1}{4}\right) = \frac{3}{16}.$$

(c) The expected value of X is

$$4 \int_0^1 \int_0^1 x^2y \, dx \, dy = \frac{4}{3} \int_0^1 x^3y \Big|_{x=0}^1 \, dy = \frac{4}{3} \int_0^1 y \, dy = \frac{4}{6}y^2 \Big|_{y=0}^1 = \frac{2}{3}.$$

The expected value of Y is

$$4 \int_0^1 \int_0^1 xy^2 dx dy = \frac{4}{2} \int_0^1 x^2 y^2 \Big|_{x=0}^1 dy = \frac{4}{2} \int_0^1 y^2 dy = \frac{4}{6} y^3 \Big|_{y=0}^1 = \frac{2}{3}.$$

16.6: #12 If we solve the equation for the sphere for z , we get

$$z = 2 \pm \sqrt{4 - x^2 - y^2}.$$

If we choose the $+$ sign and differentiate, we get

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{4 - x^2 - y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{4 - x^2 - y^2}}.$$

The sphere and paraboloid intersect in the plane $z = 3$, and their intersection is the trace of either surface in this plane, which is a circle of radius $\sqrt{3}$. The part of the sphere that lies inside the paraboloid is thus the part which sits above the disk of radius $\sqrt{3}$, which we denote $D(\sqrt{3})$. The surface area of this part of the sphere is thus

$$\iint_{D(\sqrt{3})} \sqrt{\frac{x^2}{4 - x^2 - y^2} + \frac{y^2}{4 - x^2 - y^2} + 1} dA = \iint_{D(\sqrt{3})} \sqrt{\frac{4}{4 - x^2 - y^2}} dA.$$

Changing to polar coordinates, this becomes

$$2 \int_0^{2\pi} \int_0^{\sqrt{3}} r \sqrt{\frac{1}{4 - r^2}} dr d\theta = - \int_0^{2\pi} 2\sqrt{4 - r^2} \Big|_{r=0}^{\sqrt{3}} d\theta = -2 \int_0^{2\pi} (1 - 2) d\theta = 4\pi.$$

16.6: #22 Let $0 < t < a$. The surface area of the piece of the sphere of radius a above the disk of radius t is

$$\iint_{D(t)} \sqrt{\frac{x^2}{a^2 - x^2 - y^2} + \frac{y^2}{a^2 - x^2 - y^2} + 1} dA = \iint_{D(t)} \sqrt{\frac{a^2}{4 - x^2 - y^2}} dA.$$

Changing to polar coordinates, this becomes

$$a \int_0^{2\pi} \int_0^t r \sqrt{\frac{1}{a^2 - r^2}} dr d\theta = -\frac{a}{2} \int_0^{2\pi} 2\sqrt{a^2 - r^2} \Big|_{r=0}^t d\theta = -a \int_0^{2\pi} (\sqrt{a^2 - t^2} - a) d\theta = -2a\pi(\sqrt{a^2 - t^2} - a).$$

Taking the limit of this expression as $t \rightarrow a^-$, we get that the area of the top half of a sphere of radius a is $2a^2\pi$. Thus the radius of the entire sphere is twice that, or $4a^2\pi$.

16.7: #14 The relevant inequalities here are

$$0 \leq z \leq x, \quad x^2 \leq y \leq x \quad 0 \leq x \leq 1.$$

The integral then becomes

$$\int_0^1 \int_{x^2}^x \int_0^x (x+2y) dz dy dx = \int_0^1 \int_{x^2}^x x(x+2y) dy dx = \int_0^1 (x^2 y + xy^2) \Big|_{y=x^2}^x dx = \int_0^1 (x^3 + x^3 - x^4 - x^5) dx = \frac{2}{15}.$$

16.7: #16 If we first integrate with respect to x , then we have that the integral is

$$\iint_D \int_0^{y/3} z \, dx \, dA = \iint_D \frac{yz}{3} \, dA$$

where D is the quarter disk of radius 3 in the yz plane which lies in the first quadrant. To perform this area integral, we can go to polar coordinates. The integral then becomes

$$\frac{1}{3} \int_0^{\pi/2} \int_0^3 r^3 \cos \theta \sin \theta \, dr \, d\theta = \frac{1}{12} \int_0^{\pi/2} r^4 \cos \theta \sin \theta \Big|_{r=0}^3 \, d\theta = -\frac{81 \cos^2 \theta}{12 \cdot 2} \Big|_{\theta=0}^{\pi/2} = \frac{81}{24}.$$

16.7: #34

$$\begin{aligned} \int_0^1 \int_0^{x^2} \int_0^y f(x, y, z) \, dz \, dy \, dx &= \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x, y, z) \, dz \, dx \, dy \\ &= \int_0^1 \int_y^1 \int_{\sqrt{y}}^1 f(x, y, z) \, dx \, dz \, dy = \int_0^1 \int_0^z \int_{\sqrt{y}}^1 f(x, y, z) \, dx \, dy \, dz \\ &= \int_0^1 \int_{\sqrt{z}}^1 \int_0^{x^2} f(x, y, z) \, dy \, dx \, dz = \int_0^1 \int_0^{x^2} \int_0^{x^2} f(x, y, z) \, dy \, dz \, dx. \end{aligned}$$