MATH 255 **Applied Honors Calculus III** **Winter 2011**

Homework 9

Due: Moday, April 4, 2011

Section 16.8, pg. 1073: 33, 38.
Section 16.9, pg. 1084: 3, 5, 7 13, 17, 21.
Section 17.1, pg. 1096: 15-18, 29-32.
Section 17.2, pg. 1107: 5, 7,11, 31, 46.

16.8, #33
The domain of the triple integral is the solid confined by the paraboloid $z = x^2 + y^2$ and $z = 2 - x^2 - y^2$. When we convert a triple integral from rectangular to cylindrical coordinates, we write $x = r\cos\theta, y = r\sin\theta$, and $z = z$, and replacing dV by $rdzdrd\theta$.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{x^2+y^2}^{2-x^2-y^2}(x^2+y^2)^{\frac{3}{2}}\,dzdydx = \int_0^{2\pi}\int_0^1\int_{r^2}^{2-r^2}(r^3)rdzdrd\theta = \frac{8\pi}{35}$$

Solutions
16.8, #38 *Show that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}\,dx\,dy\,dz = 2\pi.$$

(The improper triple integral is defined as the limit of a triple integral over a solid sphere as the radius of the sphere increases indefinitely.)

Solution

Let I denote the improper triple integral in question. We are told that the meaning of the improper integral is:

$$I := \lim_{R\to\infty} I_R,$$

where

$$I_R := \iiint_{E(R)}\sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)}\,dV, \qquad \text{and} \qquad E(R) := \left\{(x,y,z) \mid x^2+y^2+z^2 \le R^2\right\}.$$

Since for each $R > 0$ the region of integration $E(R)$ is a solid sphere, and since the integrand only involves the distance to the origin, it makes sense to evaluate I_R using spherical coordinates. The region $E(R)$ corresponds to the inequalities $0 \le \rho \le R$, $0 \le \theta \le 2\pi$, and $0 \le \phi \le \pi$ in spherical coordinates. Also,

$$\sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)} = \rho e^{-\rho^2}$$

and

$$dV = \rho^2\sin(\phi)\,d\theta\,d\phi\,d\rho,$$

so

$$I_R = \int_0^R \int_0^\pi \int_0^{2\pi} \rho e^{-\rho^2} \rho^2 \sin(\phi)\, d\theta\, d\phi\, d\rho.$$

Evaluating the inner integral over θ gives

$$I_R = 2\pi \int_0^R \int_0^\pi \rho^3 e^{-\rho^2} \sin(\phi)\, d\phi\, d\rho \qquad \text{because} \qquad \int_0^{2\pi} d\theta = 2\pi.$$

Next, evaluating the "next inner" integral over ϕ gives

$$I_R = 4\pi \int_0^R \rho^3 e^{-\rho^2}\, d\rho \qquad \text{because} \qquad \int_0^\pi \sin(\phi)\, d\phi = 2.$$

To evaluate the remaining integral requires some kind of integration by parts. Let's proceed this way: first make the substitution $u = \rho^2$ with $2\rho\, d\rho = du$; therefore

$$I_R = 2\pi \int_0^{R^2} ue^{-u}\, du.ue^{-u}\, du.$$

Now integrate by parts:

$$\begin{aligned}
I_R &= -2\pi \int_0^{R^2} u \left(\frac{d}{du} e^{-u} \right) du \\
&= -2\pi u e^{-u} \Big|_{u=0}^{u=R^2} + 2\pi \int_0^{R^2} e^{-u}\, du \\
&= -2\pi R^2 e^{-R^2} + 2\pi \int_0^{R^2} e^{-u}\, du \\
&= -2\pi R^2 e^{-R^2} - 2\pi e^{-u} \Big|_{u=0}^{u=R^2} \\
&= -2\pi R^2 e^{-R^2} - 2\pi e^{-R^2} + 2\pi.
\end{aligned}$$

Taking the limit $R \to \infty$ everything disappears except for the 2π:

$$I = \lim_{R\to\infty} I_R = 2\pi.$$

16.9, #3

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{v}{u+v)^2} & \frac{-u}{(u+v)^2} \\ \frac{-v}{(u-v)^2} & \frac{u}{(u-v)^2} \end{vmatrix} = 0$$

16.9 # 5

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix} = 2uvw$$

16.9 #7 Here S is a rectangle, whose boundary can be described by four lines, we will transform each line to new space, and the region confined by the transformed line can determine the new image after the transformation.

1) line one: $v = 0, 0 \le u \le 3$
$\Rightarrow x = 2u + 3v = 2u, y = u - v = u \Rightarrow y = \frac{x}{2}$ and $0 \le x \le 6$
2) line two: $u = 3, 0 \le u \le 2$
$\Rightarrow x = 6 + 3v, y = 3 - v \Rightarrow y = 5 - \frac{x}{3}$ and $6 \le x \le 12$
3) line three: $v = 2, 0 \le u \le 3$
$\Rightarrow x = 2u + 6, y = u - 2 \Rightarrow y = \frac{x}{2} - 5$ and $6 \le x \le 12$
4) line four: $u = 0, 0 \le v \le 2$
$\Rightarrow x = 3v, y = -v \Rightarrow y = -\frac{x}{3}$ and $0 \le x \le 6$

16.9, #13 1) The original domain of integral is bounded by the ellipse $9x^2 + 4y^2 = 36$. After the transformation by $x = 2u, y = 3v$, the domain change to the region bounded by
$9(2u)^2 + 4(3v)^2 = 36$, which is equal to $u^2 + v^2 = 1$. So the new domain becomes a unit circle.
2) The function x^2 has been transformed to $4u^2$.
3)

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} = 6$$

so we replace $dxdy$ by $6dudv$
4) So

$$\int\int_R x^2 dxdy = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} 4u^2 6dudv = \int_{-1}^{1} \int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} 24u^2 dudv$$

Then transform to polar coordinate by $u = r\cos\theta, v = r\sin\theta$

$$\int_0^{2\pi} \int_0^1 24(r\cos\theta)^2 r dr d\theta = \int_0^{2\pi} \int_0^1 24r^3 \cos^2\theta dr d\theta = \int_0^{2\pi} 6\cos^2\theta d\theta = 6\pi$$

16.9, #17

(a) *Evaluate* $\iiint_E dV$, *where* E *is the solid enclosed by the ellipsoid* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. *Use the transformation* $x = au$, $y = bv$, $z = cw$.

(b) *The Earth is not a perfect sphere; rotation has resulted in flattening at the poles. So the shape can be approximated by an ellipsoid with* $a = b = 6378$ *km and* $c = 6356$ *km. Use part (a) to estimate the volume of the Earth.*

For part (a), we note that the given transformation maps the solid enclosed by the ellipsoid into the solid enclosed by the unit sphere: $D = \{(u, v, w) \mid u^2 + v^2 + w^2 \le 1\}$. To find out how the volume element $dV(x, y, z)$ is written in terms of the transformed coordinates (u, v, w), we need to calculate the Jacobian:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix} = \det \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} = abc.$$

The formula for the volume element is

$$dV(x, y, z) = \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| dV(u, v, w)$$

(note that it is the absolute value of the Jacobian that arises in the formula). Assuming that a, b, and c are all positive, we therefore get

$$dV(x,y,z) = abc\, dV(u,v,w).$$

The volume contained inside the ellipsoid is therefore

$$V = \iiint_E dV(x,y,z) = \iiint_D \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dV(u,v,w) = abc \iiint_D dV(u,v,w).$$

Since the remaining integral is just another way of writing the volume of the unit sphere, we get

$$V = \frac{4}{3}\pi abc.$$

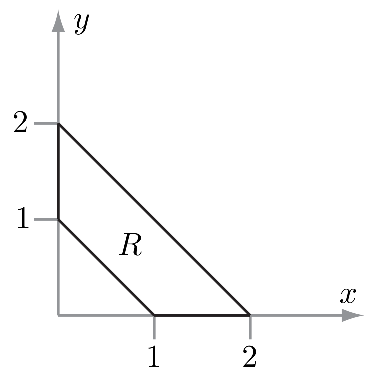
For part (b), we just evaluate this formula for the given values of a, b, and c:

$$\text{Volume of Earth} \approx \frac{4}{3}\pi 6378^2(6356)\,\text{km}^3 = 1.08 \times 10^{12}\,\text{km}^3.$$

16.9, #21 *Evaluate the integral by making an appropriate change of variables:*

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$$

where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.
The region of integration in the (x,y)-plane is shown below:



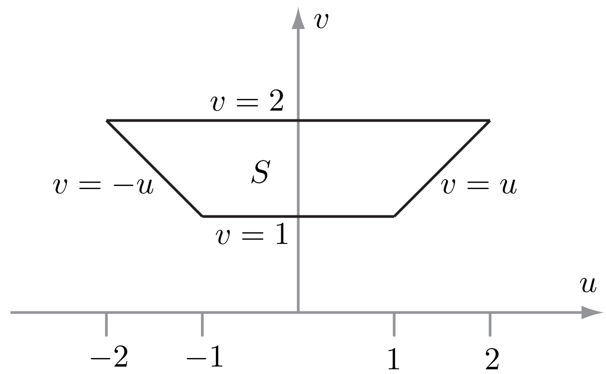
If we look at the integrand, we see that it involves the combinations of variables $u = y - x$ and $v = y + x$, and moreover, the two diagonal boundaries of the region R correspond to constant values of v, namely $v = 1$ and $v = 2$. So this suggests that we should try to rewrite the integral in terms of the transformation

$$u = y - x \quad \text{and} \quad v = y + x.$$

Equivalently, we may solve for x and y in terms of u and v:

$$x = \frac{v-u}{2} \quad \text{and} \quad y = \frac{v+u}{2}.$$

The vertical segment $x = 0$, $1 \leq y \leq 2$ then corresponds to a segment of the line $v = u$, with $1 \leq u \leq 2$. Also, the horizontal segment $y = 0$, $1 \leq x \leq 2$ corresponds to a segment of the line $v = -u$, with $-2 \leq u \leq -1$. Finally, we noted that the remaining segments of the boundary of R are mapped to segments of the lines $v = 1$ and $v = 2$, so we may assemble this information to come up with a picture of the image S of the trapezoid R in the (u,v)-plane:



Now it is easy to see that we could integrate in the (u,v)-plane by taking the outer integral to be over v with $1 \le v \le 2$, while the inner integral over u will have limits $-v \le u \le v$, and the integrand will take the form $\cos(u/v)$.
It only remains to write the area element $dA(x,y)$ in terms of u and v, so we need the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u y_v - x_v y_u = -\frac{1}{2}\cdot\frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} = -\frac{1}{2}.$$

The area element is therefore

$$dA(x,y) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA(u,v) = \frac{1}{2} dA(u,v).$$

So, as an iterated integral in the (u,v)-plane, we have

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA(x,y) = \iint_S \cos\left(\frac{u}{v}\right) \frac{1}{2}\, dA(u,v) = \frac{1}{2}\int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right) du\, dv.$$

Now, to evaluate the inner integral, we have

$$\int_{-v}^{v} \cos\left(\frac{u}{v}\right) du = v\sin\left(\frac{u}{v}\right)\Big|_{u=-v}^{u=v} = v\sin(1) - v\sin(-1) = 2v\sin(1).$$

(We used the fact that $\sin(-x) = -\sin(x)$, that is, $\sin(x)$ is an odd function of x.) Substituting this result into the iterated integral leaves just the outer integral:

$$\iint_R \cos\left(\frac{y-x}{y+x}\right) dA(x,y) = \sin(1)\int_1^2 v\, dv = \sin(1)\frac{1}{2}v^2\Big|_1^2 = \frac{3}{2}\sin(1).$$

It is worth commenting here that in this problem as in problem #17 the Jacobian turned out to be a constant, but in general it is a function of the new variables (u,v,w) or (u,v) whose absolute value must be included in the integrand.

17.1 #15-18

15: IV. Constant vectors everywhere;
16: I. Constant vectors if z is fixed;
17: III. Even x,y are zeros, vectors has certain length;
18: II. Length of vectors increase as x,y,z increase from 0.

17.1 #29-32
29:IV. $< y, x >$;
30:III. $< 2x, -2y >$;
31:II. $< 2x, 2y >$;
32:I. $\frac{<x,y>}{\sqrt{x^2+y^2}}$.

17.2 # 5
C is the parabola $y = x^2$ from (1,1) to (3,9). This can be parameterized as $\vec{r}(t) = \langle t, t^2 \rangle$ with $1 \le t \le 3$.

$$\begin{aligned}
\int_C (xy + \ln(x))dy \quad &= \textstyle\int_a^b (x(t)y(t) + \ln(x(t)))y'(t)\,dt \\
&= \textstyle\int_1^3 (t \cdot t^2 + \ln(t))2t\,dt \\
&= 2\textstyle\int_1^3 (t^4 + t\ln(t))dt \\
&= 2\left[t^5/5\right]_1^3 + 2\left[t^2/4(2\ln(t) - 1)\right]_1^3 \\
&= \tfrac{464}{5} + \tfrac{9\ln(9)}{2} = \tfrac{464}{5} + 9\ln(3)
\end{aligned}$$

17.2 # 7
There are two segments: $\vec{r_1}(t) = \langle 2t, 0 \rangle$ and $\vec{r_2}(t) = \langle 2, 3t \rangle$.

$$\begin{aligned}
\int_C xy \text{ dx} \quad &= \textstyle\int_{C_1} xy \text{ dx} + \int_{C_2} xy \text{ dx} \\
&= \textstyle\int_0^1 x_1(t)y_1(t)x_1'(t)dt + \int_0^1 x_2(t)y_2(t)x_2'(t)dt \\
&= \textstyle\int_0^1 (2t)(0)(2)dt + \int_0^1 (2)(3t)(0)dt \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\int_C (x-y) \text{ dy} \quad &= \textstyle\int_{C_1} (x-y) \text{ dy} + \int_{C_2} (x-y) \text{ dy} \\
&= \textstyle\int_0^1 (x_1(t) - y_1(t))y_1'(t)dt + \int_0^1 (x_2(t) - y_2(t))y_2'(t)dt \\
&= \textstyle\int_0^1 (2t - 0)(0)dt + \int_0^1 (2 - 3t)(3)dt \\
&= \textstyle\int_0^1 (6 - 9t)dt \\
&= 3/2
\end{aligned}$$

$\int_C xy \text{ dx} + (x-y) \text{ dy} = \int_C xy \text{ dx} + \int_C (x-y) \text{ dy} = 0 + 3/2 = 3/2$
17.2 # 11

C is just the line $\vec{r}(t) = \langle t, 2t, 3t\rangle$ with $0 \le t \le 1$. $\mid r'(t) \mid = \sqrt{1^2+2^2+3^2}$.

$$\begin{aligned}\int_C xe^{yz}ds &= \textstyle\int_a^b x(t)e^{y(t)z(t)} \mid r'(t) \mid dt \\ &= \textstyle\int_0^1 te^{2t\cdot 3t}\sqrt{14}dt \\ &= \sqrt{14}\textstyle\int_0^1 te^{6t^2}dt \\ &= \sqrt{14}/12\textstyle\int_0^6 e^u du \\ &= \sqrt{14}/12(e^6-1)\end{aligned}$$

17.2 # 31

The half-circle is parameterized as $\vec{r}(t) = 2\langle \cos(t), \sin(t)\rangle$ with $-\pi/2 \le t \le \pi/2$. So $\mid \langle r'(t)\rangle \mid = 2$.

$$m = \int_C \rho(x,y)ds = \int_{-\pi/2}^{\pi/2} 2kdt = 2k\pi$$

$$M_y = \int_C \rho(x,y)xds = \int_{-\pi/2}^{\pi/2} 2k2\cos(t)dt = 8k$$

$$M_x = \int_C \rho(x,y)yds = \int_{-\pi/2}^{\pi/2} 2k2\sin(t)dt = 0$$

So, the center of mass is at $(M_y/m,\, M_x/m) = (8/\pi,\, 0)$ and the mass is $2k\pi$.

17.2 # 46

A circle of radius r is parameterized as $\vec{r}(t) = r\langle \cos(t), \sin(t)\rangle$, and $\mid \vec{r}'(t) \mid = r$. Thus

$$\int_C \vec{B}\cdot d\vec{r} = \int_C |B|\cos\theta ds$$

where θ is the angle between $\vec{B}$ and the differential tangent vector $d\vec{r}$. The text reveals that the magnetic field is oriented along the circle, though, so $\theta = 0$

$$\int_C \vec{B}\cdot d\vec{r} = \int_C |B|ds = \int_0^{2\pi} Brdt$$

The text also hints that $|B|$ depends only on the radius from the wire.

$$\int_0^{2\pi} |B|rdt = |B|r\int_0^{2\pi} dt = 2\pi|B|r = \mu_0 I$$

Solving for $|B|$ gets $|B| = \frac{\mu_0 I}{2\pi r}$.