Homework 10

Due: Monday, April 11, 2011

Section 17.3, pg. 1117: 3, 9, 11, 29, 33.
Section 17.4, pg. 1125: 3, 9, 13, 25
Section 17.5, pg. 1132: 3, 12, 13
Section 17.6, pg. 1142: 3, 17, 21, 35, 45.

Solutions
17.3 # 3
Clear the derivatives are continuous, so check Clairaut's theorem. $\frac{\partial P}{\partial y} = 5$ and $\frac{\partial Q}{\partial x} = 5 = \frac{\partial P}{\partial y}$. So it is conservative. The potential function is $f(x, y) = 3x^2 + 5xy + 2y^2 + k$, where k is some constant.
17.3 # 9
$\frac{\partial P}{\partial y} = e^x + \cos y$ and $\frac{\partial Q}{\partial x} = e^x + \cos y = \frac{\partial P}{\partial y}$. So it is conservative. The potential function is $f(x, y) = ye^x + x \sin y + k$, where k is some constant.
17.3 # 11
a) $\vec{F}(x, y)$ is conservative: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$. Thus, it is path independent.
b) We don't have to use any of the given paths. Take the line $\vec{r}(t) = \langle 1 + 2t, 2 \rangle$ with $0 \le t \le 1$.

$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \textstyle\int_C P dx + \int_C Q dy \\ &= \textstyle\int_0^1 2x(t)y(t)x'(t)dt + 0 \\ &= \textstyle\int_0^1 2(1 + 2t)2 \cdot 2dt \\ &= 8 \textstyle\int_0^1 (1 + 2t)dt \\ &= 8(1 + 1) \\ &= 16\end{aligned}$$

17.3 # 29
It's all in one piece, so it is connected. There are no holes, so it is simply-connected. The bounds (either 0 or infinity) are not part of the domain itself, so it is open.
17.3 # 33
a) $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2}$
b) Both curves are given by $\vec{r}(t) = \langle \cos t, \sin t \rangle$. The top half has $0 \le t \le \pi$, and the bottom by $\pi \le t \le 2\pi$. Note: for C_2, we have to progress from (1,0) to (-1,0), so the limits I've given are actually for $-C_2$.
Top path:

$$\begin{aligned}\int_{C1} \vec{F} \cdot d\vec{r} &= \textstyle\int_0^\pi P dx + \int_0^\pi Q dy \\ &= \textstyle\int_0^\pi \frac{-\sin t}{\cos t^2 + \sin t^2}(-\sin t)dt + \int_0^\pi \frac{\cos t}{\cos t^2 + \sin t^2}(\cos t)dt \\ &= \textstyle\int_0^\pi 1 dt \\ &= \pi\end{aligned}$$

Bottom path:

$$\int_{-C2} \vec{F} \cdot d\vec{r} = \int_{\pi}^{2\pi} Pdx + \int_{\pi}^{2\pi} Qdy$$
$$= \int_{\pi}^{2\pi} \frac{-\sin t}{\cos t^2 + \sin t^2}(-\sin t)dt + \int_{\pi}^{2\pi} \frac{\cos t}{\cos t^2 + \sin t^2}(\cos t)dt = \int_{\pi}^{2\pi} 1dt = \pi$$
$$= -\int_{-C2} \vec{F} \cdot d\vec{r}$$

So $-\int_{C2} \vec{F} \cdot d\vec{r} = \int_{C1} \vec{F} \cdot d\vec{r} = \pi$. Not path independent.
This is not a contradiction of Theorem 6, since P and Q do not have continuous first order derivatives. They are discontinuous at (0, 0).

17.4 # 3

a) The line integral has three line segments, $\vec{r}_1(t) = \langle t, 0 \rangle$, $\vec{r}_2(t) = \langle 1, 2t \rangle$, $\vec{r}_3(t) = \langle 1-t, 2(1-t) \rangle$. Each will have $0 \le t \le 1$.

$$\int_C xydx = \int_C x(t)y(t)x'(t)dt$$
$$= \int_{C_1} x(t)y(t)x'(t)dt + \int_{C_2} x(t)y(t)x'(t)dt + \int_{C_3} x(t)y(t)x'(t)dt$$
$$= \int_0^1 t(0)(1)dt + \int_0^1 1(2t)(0)dt + \int_0^1 (1-t)2(1-t)(-1)dt$$
$$= 0 + 0 - 2\int_0^1 (1-t)^2 dt$$
$$= -2/3$$

$$\int_C x^2y^3dy = \int_C x(t)^2y(t)^3y'(t)dt$$
$$= \int_{C_1} x(t)^2y(t)^3y'(t)dt + \int_{C_2} x(t)^2y(t)^3y'(t)dt + \int_{C_3} x(t)^2y(t)^3y'(t)dt$$
$$= \int_0^1 (1)^2(0)^3(0)dt + \int_0^1 (1)^2(2t)^3(2)dt + \int_0^1 (1-t)^2 2^3(1-t)^3(-2)dt$$
$$= 0 + 2^4\int_0^1 t^3dt - 2^4\int_0^1 (1-t)^5 dt$$
$$= 4 - 8/3$$

So $\int_C xydx + x^2y^3dy = -2/3 + 4 - 8/3 = 2/3$.

b)

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \int\int_D (2xy^3 - x)dA$$
$$= \int_0^1 \int_0^{2x} (2xy^3 - x)dydx$$
$$= \int_0^1 \left(\frac{2^5x^5}{4} - 2x^2\right)dx$$
$$= 2/3$$

Of course, they're the same.

17.4 # 9

$$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy = \int\int_D \frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1) dy dx$$
$$= \int_0^1 (\sqrt{x} - x^2) dx$$
$$= 1/3$$

17.4 # 13

Note that the path as given is negatively oriented. If we use Green's Theorem, we have to remember to switch the sign. $\int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r} = \int \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA$.

$$\int \int_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \int_0^\pi \int_0^{\sin x} (2x - 3y^2) \text{dy dx}$$
$$= \int_0^\pi (2x \sin x - \sin x^3) \text{dx}$$
$$= 2 [\sin x - x \cos x]_0^\pi - [-1/3 \sin x^2 \cos x]_0^\pi - 2/3 \int_0^\pi \sin x \text{dx}$$
$$= 2(0 - 0 + \pi - 0) + 0 + 2/3(-1 - 1)$$
$$= 2\pi - 4/3$$

So $\int_C \vec{F} \cdot d\vec{r} = 4/3 - 2\pi$.

17.4 # 25

The definition of the moments are $I_y = \int \int_D x^2 \rho(x,y) dA$ and $I_x = \int \int_D y^2 \rho(x,y) dA$. Since ρ is constant, these become $I_y = \rho \int \int_D x^2 dA$ and $I_x = \rho \int \int_D y^2 dA$. If we can find $\vec{F}(x,y) = \langle P, Q \rangle$ such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial x} = x^2$ for I_y, then we can use Green's Theorem. Choose $\vec{F} = \langle 0, x^3/3 \rangle$. So

$$I_y = \rho \int \int_D x^2 dA = \rho \int_C \langle 0, x^3/3 \rangle \cdot d\vec{r} = \rho/3 \int_C x^3 \cdot dy$$

Similarly, for I_x, we choose $\vec{F} = \langle -y^3, 0 \rangle$.

$$I_x = \rho \int \int_D y^2 dA = \rho \int_C \langle -y^3/3, 0 \rangle \cdot d\vec{r} = -\rho/3 \int_C y^3 \cdot dx$$

17.5 # 3

$$\vec{\nabla} \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle 1, x + yz, xy - \sqrt{z} \rangle$$
$$= 0 + z - \frac{1}{2\sqrt{z}}$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ 1 & x + yz & xy - \sqrt{z} \end{vmatrix} = \hat{i}(x - y) - \hat{j}(y - 0) + \hat{k}(1 - 0)$$

$$\vec{\nabla} \times \vec{F} = \langle x - y, -y, 1 \rangle$$

17.5 # 12

a) Meaningless, since curl operates on vectors.
b) Vector field
c) Scalar field
d) Vector field
e) Meaningless, since gradient operates on scalar fields

f) Vector field
g) Scalar field
h) Meaningless, since divergence operates on vector fields
i) Vector field
j) Meaningless, since divergence operates on vector fields, but the first divergence would return a scalar field
k) Meaningless. The cross product works on vectors, but div($\mathbf{F}$) is a scalar.
l) Scalar field. Actually, 0, but that's a scalar.

17.5 # 13

If $\vec{\nabla} \times \vec{F} = \vec{0}$, then $\vec{F}$ is a conservative field. The cross product is $\langle (x-x), (-y+y), (z-z) \rangle = \vec{0}$. A good potential function is $f(x,y,z) = xyz$.

17.6 # 3

For a given value of x, this is clearly a circle in y and z. This is independent of x, so this surface is a cylinder along the x-axis.

17.6 # 17

$\vec{r}(u,v) = \langle 1, 2, -3 \rangle + u\langle 1, 1, -1 \rangle + v\langle 1, -1, 1 \rangle$.

17.6 # 21

Spherical coordinates are best: $\rho = 2$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi/4$. This would give $\vec{r}(\theta, \phi) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$.
You could do this in rectangular coordinates, too: Find the limits from $z = \sqrt{4 - x^2 - y^2} = \sqrt{x^2 + y^2}$ gives $x^2 + y^2 = 2$ for the intersection. But, any circular radius less than this is legal, so we get $z = \sqrt{4 - x^2 - y^2}$, $x^2 + y^2 \le 2$.

17.6 # 35

S is given by the graph $z = g(x,y) = 4 - x - 2y$ such that $x^2 + y^2 \le 4$.

$$\begin{aligned} A(S) = \int\int_S dS &= \int\int_D \sqrt{1 + g_x^2 + g_y^2}\, dA \\ &= \int\int_D \sqrt{1 + 1^2 + 2^2}\, dA \\ &= \sqrt{6} \int\int_D dA \\ &= \sqrt{5} \int_0^{2\pi} \int_0^2 \text{r dr d}\theta \\ &= 4\pi\sqrt{6} \end{aligned}$$

17.6 # 45

$\vec{r}(u,v) = \langle xu, u+v, u-v \rangle$. $\vec{r_u} = \langle v, 1, 1 \rangle$. $\vec{r_v} = \langle u, 1, -2 \rangle$.
$\int\int_S 1 dS = \int\int_D \mid \vec{r_u} \times \vec{r_v} \mid dA$.

$$\vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v & 1 & 1 \\ v & 1 & -1 \end{vmatrix} = \hat{i}(-1-1) - \hat{j}(-v-u) + \hat{k}(u-v) = \langle -2, u+v, u-v \rangle$$

$\mid \vec{r_u} \times \vec{r_v} \mid = \sqrt{4 + u^2 + v^2 + 2uv + u^2 + v^2 - 2uv} = \sqrt{4 + 2(u^2 + v^2)}$.
$\int\int_S 1 dS = \int\int_D \sqrt{4 + 2(u^2+v^2)} dA$.
Since D is a circle in u and v, this integral is easy in polar coordinates. $u = r\cos\theta$. $v = r\sin\theta$.
$\int\int_D \sqrt{4 + 2(u^2+v^2)} dA = \int_0^{2\pi} \int_0^1 (\sqrt{4 + 2r^2}) r dr d\theta$
Make a substitution, $t = 4 + 2r^2$, $dt = 4rdr$.
$\int\int_S 1 dS = 2\pi/4 \int_4^6 \sqrt{t} dt = \pi/3(6^{3/2} - 8)$