MATH 255 **Applied Honors Calculus III** **Winter 2011**

Homework 11

Due: Monday, April 18, 2011

Section 17.7, pg. 1155: 5, 13, 19, 24.
Section 17.8, pg. 1161: 3, 7, 13, 17
Section 17.9, pg. 1168: 3, 7, 19, 25.

17.7 # 5
$z = g(x,y) = 1 + 2x + 3y$.

$$\begin{aligned}\int\int_S x^2 y dS &= \int\int_D x^2 yz\sqrt{1+g_x^2+g_y^2}dA \\ &= \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{1+4+9}dydx \\ &= \sqrt{14}\int_0^3\int_0^2(x^2y+2x^3y+3x^2y^2)dydx \\ &= \sqrt{14}\int_0^3(x^2y^2/2+x^3y^2+3x^2y^3)|_0^2dx \\ &= \sqrt{14}\int_0^3(10x^2+4x^3)dx \\ &= 171\sqrt{14}\end{aligned}$$

17.7 # 13
$\int\int_S(x^2z+y^2z)dS$.
S is a hemisphere, so spherical coordinates seem reasonable. $\rho = 2$, $0 \le \phi \le \pi/2$, $0 \le \theta \le 2\pi$.
$\vec{r}(u,v) = \langle 2\sin\phi\cos\phi, 2\sin\phi\sin\theta, 2\cos\phi\rangle$.
In spherical coordinates, $\mid \vec{r}_\phi \times \vec{r}_\theta \mid = \rho^2\sin\phi = 4\sin\phi$.

$$\begin{aligned}\int\int_S(x^2z+y^2z)dS &= \int\int_D(4\sin^2\phi\cos^2\theta 2\cos\phi + 4\sin^2\phi\cos^2\theta 2\cos\phi)4\sin\phi d\phi d\theta \\ &= 32\int_0^{2\pi}\int_0^{\pi/2}\sin^3\phi\cos\phi d\phi d\theta \\ &= 64\pi\int_0^{\pi/2}\sin^3\phi\cos\phi d\phi \\ &= 32\pi\int_0^1 udu \\ &= 16\pi\end{aligned}$$

The substitution was $u = \sin^2\phi$ so $du = 2\sin\phi\cos\phi$.
17.7 # 19
S: $z = 4 - x^2 - y^2$ over $0 \le x \le 1$ and $0 \le y \le 1$. The paraboloid begins to suggest cylindrical coordinates, but the domain does not have cylindrical symmetry. So we'll keep rectangular coordinates.
$\vec{F} = \langle P,Q,R\rangle = \langle xy, yz, zx\rangle$.

$$\int\int_S \vec{F}\cdot d\vec{S} = \int\int_D (-P\tfrac{\partial g}{\partial x} - Q\tfrac{\partial g}{\partial y} + R)dA$$
$$= \int_0^1\int_0^1 [-(xy)(-2x) - (yz)(-2y) + (zx)]\,dxdy$$
$$= \int_0^1\int_0^1 [-(xy)(-2x) - y(4-x^2-y^2)(-2y) + (4-x^2-y^2)x]\,dxdy$$
$$= \int_0^1\int_0^1 [4x - x^3 + 2x^2y + 8y^2 - xy^2 - 2x^2y^2 - 2y^4]\,dxdy$$
$$= 713/180$$

Section 17.7, #24 *Evaluate the surface integral* $\iint_S \vec{F}\cdot d\vec{S}$ *for the vector field* $\vec{F}(x,y,z) = xz\vec{i} + x\vec{j} + y\vec{k}$ *and the oriented surface* S *given by the hemisphere* $x^2+y^2+z^2 = 25$, $y \geq 0$, *oriented in the direction of the positive* y*-axis. In other words, find the flux of* $\vec{F}$ *across* S.

Solution. As in problem #13, we need to start by choosing a parametrization of S, and any one will do. Given a parametrization $\vec{r} = \vec{r}(u,v)$ of S, with $(u,v) \in D \subset \mathbb{R}^2$, the formula is

$$\iint_S \vec{F}\cdot d\vec{S} = \pm\iint_D \vec{F}(\vec{r}(u,v))\cdot[\vec{r}_u(u,v)\times\vec{r}_v(u,v)]\,dA(u,v),$$

where the sign is chosen so that $\pm\vec{r}_u\times\vec{r}_v$ points in the direction of desired orientation. One parametrization of the hemisphere is to take $x = u$, $z = v$, and $y = \sqrt{25-u^2-v^2}$, where the domain D is $D = \{(u,v)$ such that $u^2+v^2 \leq 25\}$. Then, $\vec{r}(u,v) = \langle u, \sqrt{25-u^2-v^2}, v\rangle$, so

$$\vec{r}_u(u,v)\times\vec{r}_v(u,v) = \left\langle 1, -\frac{u}{\sqrt{25-u^2-v^2}}, 0\right\rangle \times \left\langle 0, -\frac{v}{\sqrt{25-u^2-v^2}}, 1\right\rangle$$
$$= \left\langle -\frac{u}{\sqrt{25-u^2-v^2}}, -1, -\frac{v}{\sqrt{25-u^2-v^2}}\right\rangle.$$

It is clear by examination of the y-component that this normal vector to S points in the opposite direction desired; therefore we select the minus sign and obtain

$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_D \vec{F}(\vec{r}(u,v))\cdot[\vec{r}_u(u,v)\times\vec{r}_v(u,v)]\,dA(u,v)$$
$$= \iint_D \vec{F}(\vec{r}(u,v))\cdot\left\langle \frac{u}{\sqrt{25-u^2-v^2}}, 1, \frac{v}{\sqrt{25-u^2-v^2}}\right\rangle\,dA(u,v).$$

Finally, using the parametrization of S we have

$$\vec{F}(\vec{r}(u,v)) = \langle uv, u, \sqrt{25-u^2-v^2}\rangle,$$

so evaluating the dot product, we arrive at the double integral formula

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left[\frac{u^2v}{\sqrt{25-u^2-v^2}} + u + v\right]\,dA(u,v).$$

This double integral is going to turn out to be zero, basically because each term in the integrand is an odd function of either u or v, and the region D is symmetrical with respect to $u \to -u$ and $v \to -v$. But to confirm this we should just evaluate the integral by iterated integration as follows:

$$\iint_S \vec{F}\cdot d\vec{S} = \int_{-5}^{+5}\int_{-\sqrt{25-u^2}}^{+\sqrt{25-u^2}} \left[\frac{u^2}{\sqrt{25-u^2-v^2}} + 1\right] v\,dv\,du + \int_{-5}^{+5}\int_{-\sqrt{25-v^2}}^{+\sqrt{25-v^2}} u\,du\,dv = 0,$$

since in each case, the inner integral is identically zero because the integrand is an odd function of the integration variable while the interval of integration is symmetric about zero. Alternatively, we might choose to parametrize S by spherical coordinates in which we take the north-south axis to be the y-axis instead of the z-axis. Thus we have $x = 5\sin(v)\cos(u)$, $y = 5\cos(v)$, and $z = 5\sin(v)\sin(u)$, with longitude angle $u \in [0, 2\pi]$ and latitude angle $v \in [0, \pi/2]$, so the domain D in the (u, v)-plane is the rectangle $D = [0, 2\pi] \times [0, \pi/2]$. Then, $\vec{r}(u, v) = \langle 5\sin(v)\cos(u), 5\cos(v), 5\sin(v)\sin(u)\rangle$, so

$$\vec{r}_u(u,v) = \langle -5\sin(v)\sin(u), 0, 5\sin(v)\cos(u)\rangle \quad \text{and} \quad \vec{r}_v(u,v) = \langle 5\cos(v)\cos(u), -5\sin(v), 5\cos(v)\sin(u)\rangle,$$

and therefore a normal vector to S is given by

$$\begin{aligned}\vec{r}_u(u,v) \times \vec{r}_v(u,v) &= 25\langle -\sin^2(v)\cos(u), \sin(v)\cos(v)\cos^2(u) + \sin(v)\cos(v)\sin^2(u), \sin^2(v)\sin(u)\rangle \\ &= 25\sin(v)\langle -\sin(v)\cos(u), \cos(v), \sin(v)\sin(u)\rangle.\end{aligned}$$

Since $0 \le v \le \pi/2$, we see that the y-component of this normal vector is nonnegative, so the vector is pointing in the desired direction for the orientation of S. Therefore, for this parametrization we need to take the plus sign in the formula:

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{r} &= \iint_D \vec{F}(\vec{r}(u,v)) \cdot [\vec{r}_u(u,v) \times \vec{r}_v(u,v)]\ dA(u,v) \\ &= 25 \iint_D \sin(v)\vec{F}(\vec{r}(u,v)) \cdot \langle -\sin(v)\cos(u), \cos(v), \sin(v)\sin(u)\rangle\ dA(u,v).\end{aligned}$$

Finally, we note that in this parametrization,

$$\vec{F}(\vec{r}(u,v)) = \langle 25\sin^2(v)\sin(u)\cos(u), 5\sin(v)\cos(u), 5\cos(v)\rangle,$$

So computing the dot product in the integrand gives

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{r} &= 125 \iint_D \left[-5\sin^4(v)\sin(u)\cos^2(u) + \sin^2(v)\cos(v)\cos(u) + \sin^2(v)\cos(v)\sin(u)\right]\ dA(u,v) \\ &= 125 \int_0^{\pi/2}\int_0^{2\pi} \left[\sin^4(v)\cos^2(u)[-\sin(u)] + \sin^2(v)\cos(v)\cos(u) + \sin^2(v)\cos(v)\sin(u)\right]\ du\,dv \\ &= 125 \int_0^{\pi/2} \left[\frac{1}{3}\sin^4(v)\cos^3(u) + \sin^2(v)\cos(v)\sin(u) - \sin^2(v)\cos(v)\cos(u)\right]_{u=0}^{u=2\pi}\ dv \\ &= 125 \int_0^{\pi/2} 0\,dv \\ &= 0.\end{aligned}$$

17.8 # 3

Using Stokes's theorem means we want $\int_{\partial S} \vec{F} \cdot d\vec{r}$, $\vec{F} = \langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\rangle$. S is the top hemisphere of radius 2, so the boundary is the circle of radius 2 in the xy plane. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 0\rangle$. $\vec{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$.

$$\int\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S} \qquad = \int_{\partial S}\vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi}\vec{F}\cdot\vec{r}'(t)dt$$

$$= 2\int_0^{2\pi}(-x(t)^2e^{y(t)z(t)}\sin t + y(t)^2e^{z(t)x(t)}\cos t)dt$$

$$= 8\int_0^{2\pi}(-\cos t^2e^0\sin t + \sin t^2e^0\cos t)dt$$

$$= 8\int_0^{2\pi}(-\cos t^2\sin t + \sin t^2\cos t)dt$$

$$= \left[\cos^3 t/3 + \sin^3 t/3\right]|_0^{2\pi}$$

$$= 0$$

17.8 # 7

To use Stokes's theorem, we identify the surface, which is the part of a plane in the 1st octant. The plane goes through (1,0,0), but most importantly, has normal vector $\vec{n} = 1/\sqrt{3}\langle 1,1,1\rangle$. This is clear, since the x-, y-, and z-directions are all equivalent. The curl is $\vec{\nabla}\times\vec{F} = -2\langle z,x,y\rangle$.

$$\int_{\partial S}\vec{F}\cdot d\vec{r} \qquad = \int\int_S\vec{\nabla}\times\vec{F}\cdot d\vec{S}$$

$$= \int_0^1\int_0^{1-x}\vec{F}\cdot\langle 1,1,1\rangle \text{ dx dy}$$

$$= \int_0^1\int_0^{1-x}(-2z-2x-2y) \text{ dx dy}$$

$$= \int_0^1\int_0^{1-x}(-2(1-x-y)-2x-2y) \text{ dx dy}$$

$$= \int_0^1\int_0^{1-x}(-2) \text{ dx dy}$$

$$= -2+1 = -1$$

17.8 # 13

We want to confirm Stokes's theorem for a paraboloid $z = x^2+y^2$, $z\le 1$ and $\vec{F} = \langle y^2, x, z^2\rangle$.

a)$\int\int_S\vec{\nabla}\times\vec{F}\cdot d\vec{S}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ y^2 & x & z^2 \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(1-2y)$$

$$\vec{\nabla}\times\vec{F} = \langle 0,0,1-2y\rangle = \langle P,Q,R\rangle$$

The surface is a graph of $z = g(x, y) = x^2 + y^2$, so we can use the following approach:

$$\begin{aligned}
\int\int_S \vec{\nabla} \times \vec{F} \cdot d\vec{S} &= \int\int_D (-g_x P - g_y Q + R) dA \\
&= \int\int_D (-g_x 0 - g_y 0 + 1 - 2y) dA \\
&= \int\int_D (1 - 2y) dA \\
&= \int_0^{2\pi} \int_0^1 (1 - 2r\cos\theta) \text{r dr d}\theta \\
&= \int_0^{2\pi} \int_0^1 (r - 2r^2\cos\theta) \text{dr d}\theta \\
&= \int_0^{2\pi} (1/2 - 2/3\cos\theta) \text{d}\theta \\
&= 2\pi/2 = \pi
\end{aligned}$$

b) $\int \vec{F} \cdot d\vec{r}$
The boundary of the surface is the curve around the open end. That is $\vec{r}(t) = \langle 1 \cdot \cos t, 1 \cdot \sin t, 1 \rangle$, $0 \le \theta \le 2\pi$. So $\vec{r}$ '(t) $= \langle -\sin t, \cos t, 0 \rangle$.

$$\begin{aligned}
\int \vec{F} \cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\text{ '(t)} dt \\
&= \int_0^{2\pi} \langle y(t)^2, x(t), z(t)^2 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \\
&= \int_0^{2\pi} \langle \sin^2 t, \cos t, 1 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle dt \\
&= \int_0^{2\pi} -\sin^3 t + \cos^2 t dt \\
&= -\left[-1/3(2 + \sin^2 t)\cos t\right]_0^{2\pi} + \left[t/2 + 1/4 \sin 2t\right]_0^{2\pi} \\
&= 0 + \pi + 0 = \pi
\end{aligned}$$

The two match, $\int\int_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int \vec{F} \cdot d\vec{r}$

17.8 # 17

The work around those edges is $\int_C \vec{F} \cdot d\vec{r}$, where C is a closed curve. Clearly, given this $\vec{F}$, we can't simply integrate. We use Stokes's theorem.
The curl is $\vec{\nabla} \times \vec{F} = \langle 2y, 2z, 2x \rangle$.
We can parameterize the surface as $\vec{r}(\theta, \phi) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$.
This leads to: $\vec{r}_\phi \times \vec{r}_\theta = \langle 4\cos\theta\sin^2\phi, 4\sin\theta\sin^2\phi, 4\sin\phi\cos\phi \rangle$.

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int \int_S \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} (\vec{\nabla} \times \vec{F}) \cdot (\vec{r}_\phi \times \vec{r}_\theta) dA$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} \langle 2y, 2z, 2x \rangle \cdot \langle 4\cos\theta \sin^2\phi, 4\sin\theta \sin^2\phi, 4\sin\phi\cos\phi \rangle dA$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} \langle 4\sin\phi\sin\theta, 4\cos\phi, 4\sin\phi\cos\theta \rangle \cdot \langle 4\cos\theta \sin^2\phi, 4\sin\theta \sin^2\phi, 4\sin\phi\cos\phi \rangle dA$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} (16\cos\theta\sin\theta\sin^3\phi + 16\cos\phi\sin^2\phi\sin\theta + 16\cos\theta\sin^2\phi\cos\phi) \mathrm{d}\theta\ \mathrm{d}\phi$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} (8\sin 2\theta \sin^3\phi + 32\cos\phi\sin^2\phi\sin\theta \mathrm{d}\theta\ \mathrm{d}\phi$$
$$= \int_0^{\pi/2} (8\sin^3\phi + 32\cos\phi\sin^2\phi\ \mathrm{d}\phi$$
$$= \left[(8(-3/4\cos\phi + 1/12\cos 3\phi) + 32(\sin^3\phi/3) \right]_0^{\pi/2}$$
$$= \left[(8(-3/4\cos\phi + 1/12\cos 3\phi) + 32(\sin^3\phi/3) \right]_0^{\pi/2}$$
$$= 16$$

The table in the back of the book was used for integrating $\sin^3\phi$, and $\sin^2\phi\cos\phi$ is recognized as the derivative of $\sin^3\phi/3$.

17.9 # 3

Calculate in both ways.

a) $\int \int \int \vec{\nabla} \cdot \vec{F} dV$

$\vec{\nabla} \cdot \vec{F} = 3 + x + 2x = 3 + 3x$

$$\int \int \int_E \vec{\nabla} \cdot \vec{F} dV = \int_0^1 \int_0^1 \int_0^1 (3 + 3x)\ \mathrm{dy\ dz\ dx}$$
$$= \int_0^1 (3 + 3x)\ \mathrm{dx}$$
$$= 3 + 3/2 = 9/2$$

b) $\int \int_{\partial E} \vec{F} \cdot d\vec{S}$ The boundary of the box is the union of the six surfaces. We need a normal vector for each, but this is easy. E.g. top has $\vec{n} = \hat{k}$ to point upwards.

$$
\begin{aligned}
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= \int\int_{top} \vec{F} \cdot \hat{k} dS + \int\int_{bottom} \vec{F} \cdot (-\hat{k}) dS + \int\int_{front} \vec{F} \cdot \hat{i} dS \\
&\quad + \int\int_{back} \vec{F} \cdot (-\hat{i}) dS + \int\int_{left} \vec{F} \cdot (-\hat{j}) dS + \int\int_{right} \vec{F} \cdot \hat{j} dS \\
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= \int\int_{top} 2xz dS - \int\int_{bottom} 2xz dS + \int\int_{front} 3x dS \\
&\quad - \int\int_{back} 2x dS - \int\int_{left} xy dS + \int\int_{right} xy dS \\
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= \int\int_{top} 2x \cdot 1 dS - \int\int_{bottom} 2x \cdot 0 dS + \int\int_{front} 3 \cdot 1 dS \\
&\quad - \int\int_{back} 2 \cdot 0 dS - \int\int_{left} x \cdot 0 dS + \int\int_{right} x \cdot 1 dS \\
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= 3 + 3/2 = 9/2
\end{aligned}
$$

The same!

17.9 # 7

The divergence is $\vec{\nabla} \cdot \vec{F} = e^x \sin y - e^x \sin y + 2yz = 2yz$.
The flux is:

$$
\begin{aligned}
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= \int\int\int_E \vec{\nabla} \cdot \vec{F} dV \\
&= \int_0^2 \int_0^1 \int_0^1 2yz \text{ dx dy dz} \\
&= 2
\end{aligned}
$$

17.9 # 19

The trick here is to note that the divergence theorem relates $\int\int_{\partial E} \vec{F} \cdot d\vec{S} = \int\int\int_V \vec{\nabla} \cdot \vec{F} dV$. But, in this case, $\partial E = S_2$. We want $\int\int_S \vec{F} \cdot d\vec{S} = \int\int_{S_2} \vec{F} \cdot d\vec{S} - \int\int_{S_1} \vec{F} \cdot d\vec{S}$. The second term is easy to compute, but the first would be hard without the divergence theorem.
$\vec{F} = \langle z^2 x, y^3/3 + \tan z, x^2 z + y^2 \rangle$. Thus $\vec{\nabla} \cdot \vec{F} = z^2 + y^2 + x^2$.

$$
\begin{aligned}
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= \int\int\int_V \vec{\nabla} \cdot \vec{F} dV \\
&= \int\int\int_V (z^2 + y^2 + x^2) dV
\end{aligned}
$$

Given the hemisphere solid and the sum of square terms, spherical coordinates seem like a good choice.

$$
\begin{aligned}
\int\int_{\partial E} \vec{F} \cdot d\vec{S} &= \int\int\int_V (z^2 + y^2 + x^2) dV \\
&= \int_0^{\pi/2} \int_0^{2\pi} \int_0^1 (\rho)^2 \rho^2 \sin\phi \text{ d}\rho \text{ d}\theta \text{ d}\phi \\
&= \int_0^{\pi/2} \sin\phi d\phi \int_0^{2\pi} d\theta \int_0^1 (\rho)^4 \\
&= 2\pi/5
\end{aligned}
$$

Now for: $\int\int_{S_1} \vec{F} \cdot d\vec{S}$. The normal vector is clearly $-\hat{k}$. Recall that z = 0 on S_1.

$$\begin{aligned}
\int\int_{S_1} \vec{F} \cdot d\vec{S} &= -\int\int_{S_1} \vec{F} \cdot \hat{k} dS \\
&= -\int\int_{S_1} x^2 z + y^2 dS \\
&= -\int\int_{S_1} y^2 dS \\
&= -\int_0^1 \int_0^{2\pi} r^2 \sin^2 \theta d\theta dr \\
&= -\pi/4dr
\end{aligned}$$

$\int\int_S \vec{F} \cdot d\vec{S} = \int\int_{S_2} \vec{F} \cdot d\vec{S} - \int\int_{S_1} \vec{F} \cdot d\vec{S} = 2\pi/5 - (-\pi/4) = 13\pi/20.$

17.9 # 25

S is the boundary of V. That is $S = \partial V$. Thus S is a *closed* surface.

$\int\int_S(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int\int_{\partial V}(\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int\int\int_V \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) dV = \int\int\int_V 0 dV = 0.$

Using the identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$.