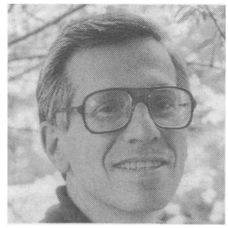
Uniqueness of Representation by Trigonometric Series

J. MARSHALL ASH[1], *DePaul University, Chicago, IL 60614*

J. MARSHALL ASH received his S.B., S.M., and Ph.D. degrees at the University of Chicago where he was a student of Antoni Zygmund. He was a Joseph Fels Ritt instructor at Columbia University from 1966 to 1969. Since then he has been on the DePaul University faculty, where he became a full professor in 1974 and chaired from 1985 to 1987. In 1977 he was visiting professor at Stanford University. His research interests have centered on multiple trigonometric series, singular integrals, and real analysis, with forays into functional and numerical analysis.



Abstract. In 1870 Georg Cantor proved that a 2π periodic complex valued function of a real variable coincides with the values of at most one trigonometric series. We present his proof and then survey some of the many one dimensional generalizations and extensions of Cantor's theorem. We also survey the situation in higher dimensions, where a great deal less is known.

1. Cantor's uniqueness theorem. In 1870 Cantor proved

THEOREM C (Cantor [5]). *If, for every real number x*

$$\lim_{N\to\infty} \sum_{n=-N}^{N} c_n e^{inx} = 0,$$

then all the complex numbers c_n, $n = 0, 1, -1, 2, -2, \ldots$ are zero.

This is called a uniqueness theorem because it has as an immediate corollary the fact that a 2π periodic complex valued function of a real variable coincides with the values of at most one trigonometric series. (Proof: Suppose $\Sigma a_n e^{inx} = \Sigma b_n e^{inx}$ for all x. Form the difference series $\Sigma(a_n - b_n)e^{inx}$ and apply Cantor's theorem.)

This theorem is remarkable on two counts. Cantor's formulation of the problem in such a clear, decisive manner was a major mathematical event, given the point of view prevailing among his contemporaries.[2] Equally enjoyable to behold is the rapid resolution that we will now sketch.

Cantor's theorem is relatively easy to prove, if, as Cantor did, you have studied Riemann's brilliant idea of associating to a general trigonometric series $T := \Sigma c_n e^{inx}$, the formal second integral, namely, $F(x) := \Sigma_{n\neq 0}(c_n/(in)^2)e^{inx} + c_0(x^2/2)$. For some interesting remarks on the importance of this idea, see the very enjoyable survey article of Zygmund [27]. Define the second Schwarz derivative D of a

[1] The research presented here was supported in part by a grant from the University Research Council of DePaul University.

The author is grateful to one referee for proposing an expanded treatment of multiple trigonometric series and to the other referee for making careful corrections and adding some historical remarks.

[2] In the eighteenth century, physicists just "did" Fourier series (often quite successfully) without worrying about convergence very much at all. When doubts about convergence began to arise in the nineteenth century, the first attempts at rigor were rather heavy handed. See Dauben [9, pp. 6–31] for an interesting discussion of the historical context.

function $G(x)$ by

$$DG(x) := \lim_{h\to 0} \frac{G(x+h) - 2G(x) + G(x-h)}{h^2}.$$

The steps of the proof are:

1. Since T converges everywhere, it is immediate that for every value of x, $c_n e^{inx} + c_{-n}e^{-inx} \to 0$ as $n \to \infty$. By the Cantor-Lebesgue theorem, $|c_n| + |c_{-n}| \to 0$ as $n \to \infty$. Appendix 1 gives Cantor's weak but easy version of this. (For a statement of the more powerful Cantor-Lebesgue theorem see the survey article by Roger Cooke [**8**]. The proof given there is much shorter than the one in Appendix 1, but requires some of the machinery of modern analysis.)

2. By the Weierstrass M-test from

$$\left| \frac{c_n e^{inx}}{(in)^2} + \frac{c_{-n}e^{-inx}}{(-in)^2} \right| \leqslant \frac{\sup(|c_n| + |c_{-n}|)}{n^2}$$

it follows that $F(x) - c_0(x^2/2)$ is a continuous function and that F is the uniform limit of its partial sums. (See Theorems 25.7 and 24.3 in [**14**] for the M-test.)

3. An important result of Schwarz's states that if G is continuous and $DG(x) = 0$ for all x, then G is a linear function. (See [**5**, pp. 82–83].) Before presenting a proof of this, Cantor remarks that Schwarz mailed this result to him from Zürich.) We give a proof in Appendix 2.

4. Since

$$\frac{e^{i(x+h)} - 2e^{ix} + e^{i(x-h)}}{h^2} = -e^{ix}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}} \right)^2$$

(Check this.), we have

$$\frac{F(x+h) - 2F(x) + F(x-h)}{h^2} = c_0 + \sum_{n\neq 0} c_n e^{inx} \left(\frac{\sin\frac{nh}{2}}{\frac{nh}{2}} \right)^2.$$

From $a_0 + \sum_{n=1}^{\infty} a_n = 0$ it follows that $\lim_{k\to 0} a_0 + \sum_{n=1}^{\infty} a_n(\sin nk/nk)^2 = 0$. See Appendix 3 for Riemann's summation by parts proof of this.

Hence $DF(x) = 0$ so $F(x) = \alpha x + \beta$ for some α and β.

5. The right side of the equation

$$c_0 \frac{x^2}{2} + \alpha x + \beta = \sum \frac{c_n}{(in)^2} e^{inx}$$

is bounded. (From 2. above it is continuous, hence bounded on $[0, 2\pi]$, and hence bounded everywhere by periodicity.) Letting $x \to \infty$ twice first shows $c_0 = 0$ and then $\alpha = 0$.

6. From the observation made in step 2 above, we see that the sequence

$$s_N(x) := -\beta + \sum_{0<|n|\leqslant N} \frac{c_n}{(in)^2} e^{inx}$$

converges uniformly to 0. But for each $n \neq 0$,

$$c_n = \frac{(in)^2}{2\pi} \int_0^{2\pi} s_N(x) e^{-inx}\, dx$$

for all $N \geqslant n$ by the orthonormality of $\{e^{inx}/\sqrt{2\pi}\}$, so letting $N \to \infty$ gives $c_n = 0$ for all $n \neq 0$. (The uniformity of convergence allows the interchange of limit and integral.) Q.E.D.

Cantor's beautiful theorem suggests a variety of extensions and generalizations. The remaining four sections of this paper will consider some of these.

2. Summability and uniqueness. Can we improve Theorem C by weakening the assumption of convergence to zero to an assumption of being merely summable to zero? As is often the case in mathematics, the starting point is a counterexample which destroys the "obvious extension."

The trigonometric series $\Sigma c_n e^{inx}$ is said to be Abel summable to s if for each r, $0 \leqslant r < 1$, $f(x, r) := \Sigma c_n e^{inx} r^{|n|}$ converges and if $\lim_{r \to 1^-} f(x, r) = s$. Let $z := re^{ix} = r(\cos x + i \sin x)$. Differentiate the identity

$$\sum_{n=0}^{\infty} (\cos nx) r^n = \mathcal{Re}\left\{\sum_{n=0}^{\infty} z^n\right\} = \mathcal{Re}\left\{\frac{1}{1-z} \cdot \frac{1-\bar{z}}{1-\bar{z}}\right\} = \frac{1 - r\cos x}{1 - 2r\cos x + r^2}$$

with respect to x to obtain

$$\sum_{n=1}^{\infty} (n \sin nx) r^n = \frac{(1-r^2) r \sin x}{(1 - 2r\cos x + r^2)^2}.$$

If $x \neq 0$, as $r \to 1^-$, the right side tends to 0; while if $x = 0$, every term of $f(0, r)$ is 0, so that $f(0, r) = 0$, whence $\lim_{r \to 1^-} f(0, r) = 0$. Thus $\Sigma_{n=1}^{\infty} n \sin nx$ is *everywhere* Abel summable to 0. Although this example is unpleasant, it turns out to be just about the worst thing that can happen.

THEOREM V (VERBLUNSKY [**25**, VOL. I, PP. 352, 383], [**22**], [**23**]). *If $c_n/|n| \to 0$ as $|n| \to \infty$ and $\Sigma c_n e^{inx}$ is Abel summable to 0 at every x, then all c_n are 0.*

3. Higher dimensions. When we move from one to several dimensions, the picture becomes much more cloudy. Here is the land of opportunity. Almost nothing is known; almost every question that the novice might ask turns out to be an open question. The first goal is to mimic Cantor's Theorem C. Even this apparently modest goal remains to a large extent unachieved. The hypothesis that $\Sigma c_n e^{inx}$ converges to 0 has several different interpretations in higher dimensions. Most of these can be illustrated in two dimensions, so to ease notation I will restrict myself to that case. The basic object will be the double trigonometric series $T(x, y) := \Sigma_{(m,n) \in \mathbf{Z} \times \mathbf{Z}} c_{mn} e^{i(mx+ny)}$. We define a rectangular partial sum of T to be

$$T_{mn}(x, y) := \sum_{|\mu| \leqslant m,\, |\nu| \leqslant n} c_{\mu\nu} e^{i(\mu x + \nu y)},$$

a diamond shaped partial sum to be

$$T^n(x, y) := \sum_{|\mu| + |\nu| \leqslant n} c_{\mu\nu} e^{i(\mu x + \nu y)},$$

and a circular partial sum to be

$$T_r(x, y) := \sum_{\mu^2+\nu^2 \leqslant r^2} c_{\mu\nu} e^{i(\mu x + \nu y)}.$$

We freeze x and y and make five different definitions of convergence.
If $T_r \to s$ as $r \to \infty$ say T is *circularly convergent* to s.
If $T^n \to s$ as $r \to \infty$ say T is *triangularly convergent* to s.
If $T_{nn} \to s$ as $r \to \infty$ say T is *square convergent* to s.
If $T_{mn} \to s$ as $\min\{m, n\} \to \infty$ say T is *unrestrictedly rectangularly convergent* to s.
If $T_{mn} \to s$ as $\min\{m, n\} \to \infty$ in such a way that m/n and n/m stay less than e, and if this happens for each (arbitrarily large) $e > 1$, say T is *restrictedly rectangularly convergent* to s.

To each of these five notions of convergence there corresponds a putative extension of Theorem C. The first of these to be proved was the following.

THEOREM SC (V. Shapiro and R. Cooke [**19**], [7]). *If $T(x, y)$ is circularly convergent to* 0 *everywhere, then all c_{mn} are* 0.

In 1957 Victor Shapiro proved a two-dimensional version of Theorem V which implied a weak version of Theorem SC that required the additional hypothesis that

$$\frac{1}{r} \sum_{(r-1)^2 < m^2+n^2 \leqslant r^2} |c_{mn}| \to 0 \quad \text{as} \quad r \to \infty. \tag{1}$$

That this hypothesis was not needed was a consequence of a generalization of the Cantor-Lebesgue theorem due to Roger Cooke in 1971. (See Theorem A1 below. A survey of Cooke's theorem and extensions of it by Zygmund [**26**] and Connes [**6**] can be found in the MONTHLY article by Cooke [**8**].)

The proof of Theorem SC is modeled after Verblunsky's Theorem V mentioned above. Both these theorems carry out the Riemann-Cantor program of integrating twice and then differentiating twice. To get at the ideas behind Theorem SC, assume that T is circularly convergent to zero everywhere. Write $M := (m, n)$, $X := (x, y)$, $M \cdot X = mx + ny$, and add the assumption that $c_{00} = 0$. (This simplifies the notation, but not the proof.)

Let

$$F(X, t) := -\sum_{M \neq 0} \frac{c_M}{|M|^2} e^{iM\cdot X - |M|t}$$

and let

$$F(X) := \lim_{t \to 0} F(X, t).$$

Then *formally*

$$F(X) \text{“} = \text{”} -\sum_{M \neq 0} \frac{c_M}{|M|^2} e^{iM\cdot X} =: S$$

and *formally* $F(X)$ is a second integral of $T(X)$.

From Cooke's two-dimensional Cantor-Lebesgue theorem it follows that equation (1) holds. This guarantees the existence of $F(X, t)$. That $F(X)$ exists for all X follows from the convergence and consequent circular Abel summability of T [**18**, pp. 67–68]. We have

$$\sum_{M \neq 0} \frac{|c_M|^2}{|M|^4} \leqslant \sum_{|M|=1} |c_M|^2 + \sum_{k=2}^{\infty} \frac{1}{(k-1)^4} \left\{ \sum_{k-1<|M| \leqslant k} |c_M|^2 \right\}$$

Condition (1) implies $|c_M| = \circ(|M|)$ so applying first this and then (1) itself shows the curly bracketed sum to be $\circ(k^2)$. Hence the sum is finite, so that by the Riesz-Fischer Theorem [**20**, p. 248] S is the Fourier series of a square integrable function. A basic fact about circular Abel summability implies that the function in question must be F [**20**, Cor. 2.15, p. 256]. In short $\{c_M/|M|^2\}$ are the Fourier coefficients of the square integrable function $F(X)$.

The first major obstacle is that in distinction to the one dimensional case, it is quite difficult to demonstrate the continuity of F.

In fact if F is continuous on the closure of an open disc B, then F is actually harmonic on B. This depends on an argument very much like the one given in appendix 2. The substitute for the second Schwarz derivative D used there is the generalized Laplacian Δ defined by

$$\Delta F(X) := \lim_{h \to 0} \frac{8}{h^2} \{ F_h(X) - F(X) \},$$

where

$$F_h(X) := \frac{1}{\pi h^2} \int_{|H| \leqslant h} F(X+H)\, dH.$$

This agrees with the usual Laplacian $(= \partial^2 F/\partial x^2 + \partial^2 F/\partial y^2)$ for C^2 functions as can be seen by expanding F into its Taylor series. We also have the representation

$$F(X,t) = \frac{t}{2\pi} \int_{R^2} \frac{F(U)}{\left[t^2 + (X-U)^2\right]^{3/2}}\, dU \quad [\mathbf{18}, \text{p. } 56].$$

Changing to polar coordinates and integrating by parts gives

$$F(X,t) = \frac{3}{2} t \int_0^{\infty} \frac{r^3 F(X)}{(t^2 + r^2)^{5/2}}\, dr \quad [\mathbf{18}, \text{pp. } 66\text{–}67]. \tag{2}$$

Now

$$\frac{d^2}{dt^2}[F(x,t)] = -\sum c_M e^{iM \cdot X - |M| t}$$

is the circular Abel means of the original series, so that from the hypothesis (1) it is immediate that $\lim_{t \to 0} (d^2/dt^2)[F(x,t)] = 0$. Shapiro then adapts a clever one-dimensional lemma of Rajchman [**25**, vol 1, pp. 353–354] to conclude from this that $\Delta F(X) = 0$ at each X. This implication depends heavily on equation (2) [**18**, pp. 66–67].

By an argument like the one in appendix 2, zero generalized Laplacian forces harmonicity. (The analog of the subtracted linear function of appendix 2 is the Poisson integral of F.) See Radó for details [**12**, p. 14]. It follows that if $F(X)$ can be shown continuous everywhere, it will be harmonic everywhere. The continuous function F will be necessarily bounded on the compact set $[0, 2\pi] \times [0, 2\pi]$ and consequently by periodicity bounded on the entire plane. But bounded harmonic functions are constant. (Apply Liouville's Theorem to the bounded analytic function $\exp(F + i\tilde{F})$, where $\tilde{F}$ is a harmonic conjugate of F.) So there will be a constant d so that

$$d + \sum_{M \neq 0} \frac{c_M}{|M|^2} e^{iM \cdot X} = 0$$

for all X. By the uniqueness theorem for square integrable functions (an immediate consequence of Parseval's formula [**25**, vol. II, p. 301]) it will follow that all $c_M = 0$.

It remains only to establish the continuity of F. This is done by generalizing a Baire category argument employed by Verblunsky.

Since $T(x, t)$ the circular Abel mean of the original series is continuous on $\{(X, t)\colon t > 0\}$ and has finite limit (namely 0) at each point, by a basic theorem of Baire for each disc in T^2 there is a subdisc and a constant K so that $|T(X, t)| \leqslant K$ for every X of the subdisc and every positive t [**25**, Vol. I, p. 29 (12.3i)] and [**18**, pp. 69–70]. Integration in the t variable then shows that in that subdisc $F(X)$ is continuous, being the uniform limit of $F(X, t)$.

Let Z be the set where F is not continuous. Let $\bar{Z}$ be the closure of Z. Assume Z is non-empty. An inspired idea that appears already in the proof of Theorem V now produces the desired contradiction, as follows. Applying the same Baire category argument that showed $F(X, t)$ well-behaved on an open dense set to the set $\bar{Z}$ produces a point $X_0 \in Z$ with $F(X, t)$ converging uniformly to $F(X)$ with respect to $\bar{Z}$ throughout a neighborhood of X_0. If X is any point very close to X_0, let X_1 be a point of $\bar{Z}$ closest to X, say $|X_1 - X| = s$. Assume $X \neq X_1$, $X_1 \neq X_0$. (If either equality holds, the argument is even easier.) Then (i) $F(X) = F_s(X)$, (ii) $F_s(X)$ is close to $F_s(X_1)$, (iii) $F_s(X_1)$ is close to $F(X_1, s)$, (iv) $F(X_1, s)$ is close to $F(X_1)$, and (v) $F(X_1)$ is close to $F(X_0)$; whence F is continuous at X_0, contrary to the definition of Z. (Reasons: (i) Continuity forces harmonicity as mentioned above, and harmonicity is equivalent to the mean value property ([**12**], p. 7). (ii), (iii) These need technical lemmas whose proofs are straightforward provided one is aware of the delicate estimate

$$\left| \int_{|X| \leqslant 1} e^{iX \cdot t} \, dX \right| \leqslant \frac{C}{|t|^{3/2}}$$

for large t. This is equivalent to the fact that the Bessel function $J_1(s) = 0(s^{-1/2})$ as $s \to +\infty$ [**18**, pp. 68–69] and [**20**, p. 199]. (iv) Uniformity of convergence gives this. (v) Uniform limits are continuous.)

This completes our discussion of the proof of Theorem SC.

The other major result in two dimensions is this.

THEOREM AW (J. M. Ash and G. Welland) [**1**], [**2**]. *If $T(x, y)$ is unrestrictedly rectangularly convergent to 0 everywhere, then all c_{mn} are 0.*

From the hypothesis it is immediate that at each X the partial sums tend to 0 "in the northeast," i.e., that

$$\lim_{\min\{m, n\} \to \infty} T_{mn}(X) = 0.$$

One-dimensional convergent sequences of numbers are necessarily bounded, but two dimensional ones need not be. Nevertheless it can be proved that at each X,

$\sup_{m,n}|T_{mn}(X)| < \infty$. In fact this is the hardest part of the proof of Theorem AW. It requires a technique that first appeared in the unpublished thesis of P. J. Cohen [2, pp. 402, 404–407].

Write

$$C_{mn}(X) := c_{m,n}e^{i(mx+ny)} + c_{-m,n}e^{i(-mx+ny)} + c_{m,-n}e^{i(mx-ny)} + c_{-m,-n}e^{i(-mx-ny)}.$$

The "Mondrian" identity

$$C_{mn}(X) = T_{mn}(X) - T_{m-1,n}(X) - T_{m,n-1}(X) + T_{m-1,n-1}(X)$$

implies that at each X the $C_{mn}(X)$ also tend to 0 in the northeast and are bounded [2, pp. 410–411]. A Cantor-Lebesgue theorem finally gets back to the coefficient themselves. The result is that $\{c_{mn}\}$ is a bounded sequence and $\lim_{\min\{|m|,|n|\}\to\infty} c_{mn} = 0$ [2, p. 408 and p. 411]. These two facts allow one to readily deduce that hypothesis (1) holds [2, p. 423]. It is also true that a unrestrictly rectangularly convergent double sequence of numbers is circularly Abel summable to the same value, *provided the partial sums are bounded* [2, pp. 413–416]. In particular, since $T(X)$ converges unrestrictly rectangularly to 0 everywhere, it is circularly Abel summable to 0 everywhere. A careful examination of the proof of Theorem SC shows that condition (1) together with everywhere circular Abel summability to 0 are sufficient hypotheses for the proof to work.

Remarks. When Grant Welland and I discovered this proof, I felt that there was an element of good luck involved here. First, the rectangular convergence gave just enough control of the coefficient size to force condition (1) and condition (1) is, in a sense, sharp. (The series $T_0(x, y) := \Sigma n \sin nx$ "almost" satisfies (1) and is circularly Abel summable to 0 everywhere. The computations are in section 2 above.) More impcrtantly, there are usually no nontrivial connections between various modes of convergence and in particular unrestricted rectangular convergence of a trigonometric series on a set does not force circular convergence of the series, even if one is willing to discard a subset of measure 0 [2, pp. 417–420]. Thus the above mentioned result connecting unrestricted rectangular convergence and circular Abel summability came as a happy surprise.

The other side of the coin is that this proof of Theorem AW was "too lucky." Often rectangular results for two-dimensional series can be extended to higher dimensions without much additional effort. The unexpected dependence of Theorem AW on Theorem SC means that a good three-dimensional uniqueness theorem for unrestricted rectangular convergence awaits either a good three-dimensional spherical uniqueness theorem or a completely new method of proof. Shapiro has proved a three-dimensional spherical uniqueness theorem, but it needs the hypothesis

$$\frac{1}{r^{\alpha}} \sum_{(r-1)^2 < l^2+m^2+n^2 \leqslant r^2} |c_{lmn}| \to 0 \quad \text{as} \quad r \to \infty \qquad (N_\alpha)$$

with $\alpha = 1$. But a three-dimensional Cantor-Lebesgue theorem can only be expected to produce condition (N_α) with $\alpha = 2$. (See [2, p. 425] for a relevant counterexample.) Thus even if one assumes that $T(x, y, z)$ is everywhere circularly convergent to 0, it is not known whether all c_{lmn} must then be 0. There is even greater ignorance concerning uniqueness questions about the other 3 modes of convergence mentioned above. For example, here are 3 open questions in two dimensions.

Question 1. If $T(x, y)$ is everywhere restrictedly rectangularly convergent to 0, does this force all c_{mn} to be 0?
Question 2. If $T(x, y)$ is everywhere square convergent to 0, does this force all c_{mn} to be 0?
Question 3. If $T(x, y)$ is everywhere triangularly convergent to 0, does this force all c_{mn} to be 0?

Again, it must be emphasized that easy counterexamples show that not much help will be available from Cantor-Lebesgue type theorems for square, restricted rectangular, or triangular convergence [**2**, pp. 416–418].

4. Fourier series. Return to one dimension. A third type of extension of Cantor's Theorem C occurs when the limit of the trigonometric series $S := \Sigma c_n e^{inx}$ is a Lebesgue integrable function f. The question now is whether S is necessarily the Fourier series of f, i.e., whether for each integer n there must hold the relation

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx.$$

A typical result in this direction is the following.

THEOREM (de la Vallée-Poussin [**25**, Vol. I, pp. 326, 382], [**21**]). *If S converges to f at each x, and if f is finite at each x and if $\int_0^{2\pi}|f(x)|\,dx < \infty$, then S is the Fourier series of f.*

5. Sets of uniqueness. If we replace the hypothesis $\Sigma c_n e^{inx} = 0$ everywhere by $\Sigma c_n e^{inx} = 0$ for x *not* in a "thin" subset E of $[0, 2\pi]$, we may still be able to conclude that all $c_n = 0$ if E is thin enough. Sufficiently thin sets E are called sets of uniqueness. A restatement of Theorem C is that the empty set is a set of uniqueness. It was already proved by W. H. Young [**24**] that every countable subset of $[0, 2\pi]$ is a set of uniqueness. (Cantor's earlier work showing that closed countable sets were sets of uniqueness led Cantor to the creation of set theory!) However, a set E of positive measure is too thick to be a set of uniqueness. (Proof from [**25**, Vol. I, p. 344]: Let E_1 be a subset of E which is perfect and of positive measure, and let $f(x)$ be the characteristic function of E_1. The Fourier series of f converges to 0 outside E_1, and so also outside E, but does not vanish identically since its constant term is $|E_1|/2\pi > 0$.) One of the most interesting theorems in all analysis produces two classes of uncountable measure zero sets which are very much like each other metrically, although the first are sets of uniqueness and the second are not.

Let $0 < \xi < 1/2$. Dissect $[0, 2\pi]$ into 2 closed "white" intervals, each of length $2\pi\xi$, $[0, 2\pi\xi]$ and $[2\pi(1-\xi), 2\pi]$, and one open "black" interval $(2\pi\xi, 2\pi(1-\xi))$. Remove the black interval and repeat the process by dissecting each white interval into 2 closed white intervals of length $2\pi\xi^2$, and 1 centered open black interval of length $2\pi\xi - 2\cdot 2\pi\xi^2$. Iterating this process k times produces 2^k closed white intervals, each of length $2\pi\xi^k$. (See FIGURE 1.)

Now let $k \to \infty$ remembering to remove the black intervals at each stage. The resulting set $E(\xi)$ is said to be of Cantor type with constant ratio of dissection.

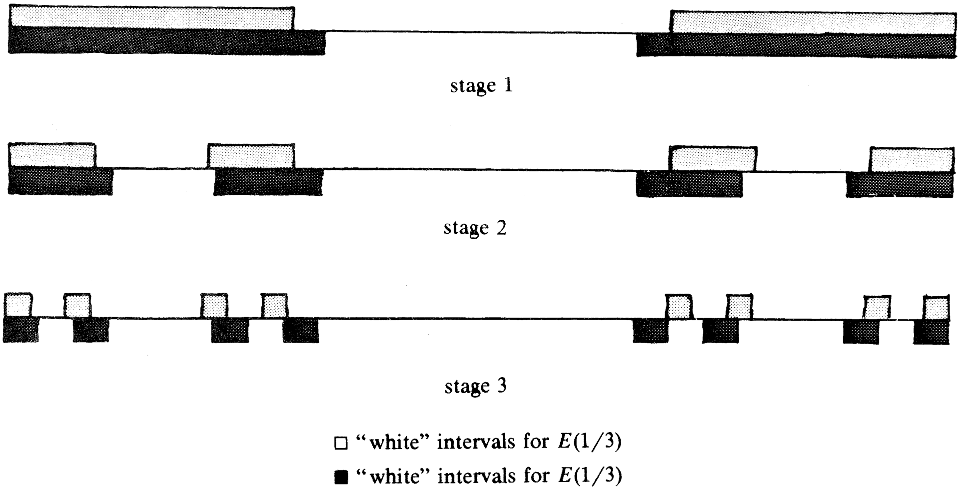


FIG. 1.

Since $2^k \cdot 2\pi\xi^k \to 0$ as $k \to \infty$, $E(\xi)$ always has measure zero. In particular $E(1/3)$ is the classical Cantor set.

THEOREM SZ (R. Salem and A. Zygmund **[16]**, **[17]**, **[25**, Vol. II, p. 152], **[15]**, **[11]**). *If $1/\xi$ is an algebraic integer all of whose conjugates have modulus less than* 1 (*i.e.*, $(1/\xi)^n + a_1(1/\xi)^{n-1} + \cdots + a_n = 0$ *for some integers* $a_1, a_2, \ldots, a_n$, *n is minimal, and the other $n-1$ roots of* $x^n + a_1x^{n-1} + \cdots + a_n = 0$ *have absolute value less than* 1), *then $E(\xi)$ is a set of uniqueness. Otherwise $E(\xi)$ is not a set of uniqueness.*

Now $1/3 = .33\ldots > .3 = 3/10$ so the process that forms $E(1/3)$ leaves more material at each stage than does the process that forms $E(3/10)$. Hence $E(1/3)$ is intuitively thicker than $E(3/10)$. Nevertheless $1/(1/3)$ satisfies $x - 3 = 0$ and hence is an algebraic integer without conjugates so that $E(1/3)$ is a set of uniqueness; while $1/(3/10)$ is not an algebraic integer at all[3] so that $E(3/10)$ is not a set of uniqueness. Thus the "wrong" one of the pair $E(1/3)$, $E(3/10)$ is the set of uniqueness.

Even more dramatically counterintuitive is the fact that the "very thick" set $\bigcup E(\xi)$, where ξ varies over all reciprocals of algebraic integers with small conjugates ($\xi = 1/2$ must be excluded), is also a set of uniqueness. This is true because N. Bary has proved that a countable union of closed sets of uniqueness is still a set of uniqueness. See ([**25**], Vol. I, p. 349) for this fact as well as further remarks on Theorem SZ.

The other side of the coin from a set of uniqueness is a set of multiplicity. A set of multiplicity by definition is a subset of $[0, 2\pi]$ which is not a set of uniqueness. More directly, E is a set of multiplicity if there is a trigonometric series which converges to 0 outside E but which does not vanish identically. Just as sets of

[3]Assume 10/3 satisfies $x^n + \cdots + a_n = 0$ with all a_i integer. Clearly $n \geqslant 2$, so $(10^{n-1} + \cdots + a_{n-1}3^{n-1})10 + a_n3^n = 0$. Hence 3 divides $10^{n-1} + 3(a_1 + \cdots + a_{n-1}3^{n-2})$, so 3 divides 10^{n-1}, a contradiction.

uniqueness become more interesting as they get "thicker", so sets of multiplicity become more interesting as they get "thinner." As was shown above, every set of positive measure is a set of multiplicity. The first measure zero set of multiplicity was produced by Men'shov [**10**] and Theorem SZ above gives lots of examples.

In a later review I will survey some of the extensive recent work that has been done on sets of uniqueness. At present this is the most active and exciting area of the four extensions I have discussed. The interested reader is encouraged to turn now to the books of Zygmund [**25**] or Bary [**3**] for much more comprehensive overviews of all the one-dimensional topics I have highlighted here.

Note added in proof. A remarkable facet of Cantor's proof of Theorem C has been *its* uniqueness. The many difficulties encountered in attempts to generalize it suggest that a different proof of Theorem C could prove useful. Because of a recent development in real analysis I can now give such a proof. Suppose the series $\Sigma c_n e^{inx}$ converges to zero everywhere. Form the *first* integral $c_0 x + \Sigma(c_n/in)e^{inx}$. Although this L^2 function is not easily seen to be continuous (The examples $\Sigma \sin n0/\ln n = 0$ but $-\Sigma \cos n0/n \ln n$ divergent give the flavor of the difficulty in working with a first integral.), it does follow directly from a theorem of Rajchman and Zygmund [**25**, Vol. 1, p. 324] that it has symmetric approximate derivative 0 at every point. By the aforementioned recent results {"A symmetric density property; monotonicity and the approximate symmetric derivative," *Proc. Amer. Math. Soc.*, 104 (1988) 1078–1102, and "A symmetric density property for measurable sets," *Real Analysis Exchange*, 14 (1988–89) 203–209, both by C. Freiling and D. Rinne}, there is a constant c so that

$$c_0 x + \sum \frac{c_n}{in} e^{inx} - c = 0$$

almost everywhere. The conclusion of Theorem C now follows from periodicity and Plancherel's Theorem. I will publish the details of this proof in the *Proc. Amer. Math. Soc.*

Appendix 1. THEOREM A1 (Cantor [**4**]). *If $c_n e^{inx} + c_{-n} e^{-inx} \to 0$ as $n \to \infty$ for every x, then $|c_n| + |c_{-n}| \to 0$.*

Proof. From the convergence to zero of $c_n e^{inx} + c_{-n} e^{-inx}$ follows the convergence to zero of its real and imaginary parts. One may easily find real numbers a_n, b_n, a_n', b_n', so that

$$c_n e^{inx} + c_{-n} e^{-inx} = (a_n \cos nx + b_n \sin nx) + (a_n' \cos nx + b_n' \sin nx)i$$

and direct calculation shows

$$a_n^2 + b_n^2 + a_n'^2 + b_n'^2 = 2(|c_n|^2 + |c_{-n}|^2).$$

Hence it suffices to prove that $a_n \cos nx + b_n \sin nx \to 0$ for every x implies $a_n^2 + b_n^2 \to 0$. Define $\rho_n := \sqrt{a_n^2 + b_n^2}$ and find θ_n so $\rho_n \cos \theta_n = a_n$, $\rho_n \sin \theta_n = b_n$. Then $a_n \cos nx + b_n \sin nx = \rho_n \cos(nx - \theta_n)$, $\rho_n \cos(nx - \theta_n) \to 0$ for every x, and we need only show that $\rho_n \to 0$.

If ρ_n does not tend to 0, there is a subsequence $\{n_k\}$ and a number δ so that for every positive integer k, $\rho_{n_k} \geqslant \delta > 0$.

By discarding as many terms as necessary from $\{n_k\}$ we may also assume $(n_{k+1}/n_k) \geqslant 3$ for all k. Then

$$\cos(n_1 x - \theta_{n_1}) \geqslant \frac{1}{2} \qquad \text{for } x \in I_1 := \left[\left(\theta_{n_1} - \frac{\pi}{3}\right)\Big/ n_1, \left(\theta_{n_1} + \frac{\pi}{3}\right)\Big/ n_1\right].^4$$

Since

$$|I_1| = \frac{2\pi}{3n_1} \quad \text{and} \quad \frac{n_2}{n_1} \geqslant 3,$$

as x ranges over I_1, $(n_2 x - \theta_{n_2})$ ranges over $n_2 I_1 - \theta_{n_2}$ and

$$|n_2 I_1 - \theta_{n_2}| = n_2 |I_1| = n_2 \cdot \frac{2\pi}{3n_1} \geqslant 2\pi.$$

Thus there is a closed interval $I_2 \subset I_1$ with $\cos(n_2 x - \theta_2) \geqslant 1/2$ for all $x \in I_2$ and $|I_2| = 2\pi/3n_2$. Proceeding inductively produces a point $\xi \in \bigcap_{k=1}^{\infty} I_k$ with $\cos(n_k \xi - \theta_{n_k}) \geqslant 1/2$ for every k. Thus $\rho_{n_k} \cos(n_k \xi - \theta_{n_k}) \geqslant \delta/2$ for every k. This contradicts the convergence to zero of $\rho_n \cos(nx - \theta_n)$ at $x = \xi$.

Appendix 2. THEOREM A2 (Schwarz, Cantor [**5**], [**9**, p. 33]. *A continuous function G with everywhere* 0 *Schwarz derivative is necessarily a linear function.*

Proof. First suppose that there is a continuous nonconvex function $H(x)$ satisfying $DH(x) > 0$. Then there are points $a < b < d$ with $H(b) > L(b)$, where L is the linear function whose graph passes through $(a, H(a))$ and $(d, H(d))$. Let $H_1(x) := H(x) - L(x)$. Then $H_1(a) = H_1(d) = 0$, $H_1(b) > 0$, H_1 is continuous and $DH_1(x) = DH(x) > 0$ for all x. Let c be a point of $[a, d]$ where H_1 is maximum. Note $c \in (a, d)$. (See Figure 2.) Then for all $h \leqslant \min\{c - a, d - c\}$, $(1/2)[H_1(c + h) + H_1(c - h)] - H_1(c) \leqslant 0$, contrary to $DH_1(c) > 0$.

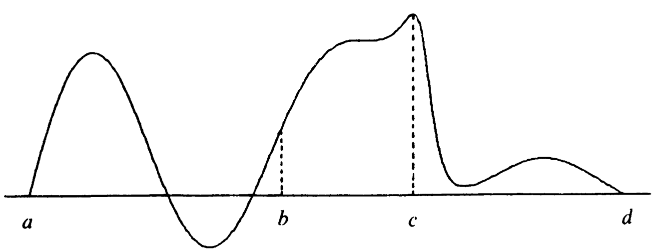


FIG. 2.

Now from $D(x^2) = 2$ (an easy calculation) and $DG = 0$, for each $\varepsilon > 0$ we have $D(G + \varepsilon x^2)(x) > 0$ for all x, so $G + \varepsilon x^2$ is convex. Let $\varepsilon \to 0$ to see that G is convex. Symmetrically, $G - \varepsilon x^2$ is concave and hence G is also. Since the graph of

[4] Warning: Here we must think of $[0, 2\pi]$ as $R/2\pi Z$, i.e. as the circumference of the unit circle. All this really means is that the set I_1 should be thought of as an interval of length $2\pi/3n_1$, even if θ_{n_1}/n_1 is less than $\pi/3n_1$ from 0 or 2π.

G lies both above and below the chord passing through any two of its points, it is a line.

Appendix 3. THEOREM A3 (Riemann [**13**, §8, Theorem 1]. *Let*

$$s_{nk} := \left(\frac{\sin nk}{nk}\right)^2, \qquad n > 0, \qquad s_{0k} := 1$$

and suppose $\sum_{n=0}^{\infty} a_n = 0$. *Let* $G_k = \sum_{n=0}^{\infty} a_n s_{nk}$. *Then* $\lim_{k\to\infty} G_k = 0$.

Proof. For each fixed $k \neq 0$, the series defining G_k converges since $a_n \to 0$ and $|s_{nk}| \leqslant c/n^2$. Summation by parts yields

$$\sum_{n=0}^{N} a_n s_{nk} = \sum_{n=0}^{N-1} s_n\left(s_{nk} - s_{n+1k}\right) + s_N s_{Nk},$$

where $s_n = a_0 + \cdots + a_n$. Let $N \to \infty$ to get

$$G_k = \sum_{n=0}^{\infty} s_n\left(s_{nk} - s_{n+1k}\right).$$

Now for each integer $N \geqslant 1$ and each $k \neq 0$,

$$|G_k| \leqslant \sup|s_n| \cdot \sum_{n=0}^{N-1}\left(|s_{nk} - 1| + |1 - s_{n+1k}|\right) + \left(\sup_{n \geqslant N}|s_n|\right)\sum_{n=N}^{\infty}|s_{nk} - s_{n+1k}|$$
$$=: A + B.$$

Since $|s_{nk} - s_{n+1k}| = |\int_{nk}^{(n+1)k} f(x)\,dx|$ where $f(x) := \{[\sin x/x]^2\}'$, the sum in B is bounded by $\int_0^\infty |f(x)|\,dx$. This is a finite constant since

$$f(x) = 2\left(\frac{\sin x}{x}\right)\left(\frac{x\cos x - \sin x}{x^2}\right),$$

$$x\cos x - \sin x \cong x\left(1 - \frac{x^2}{2}\right) - \left(x - \frac{x^3}{6}\right) = -\frac{x^3}{3}$$

near $x = 0$ so that f is bounded on $(0,1]$, and $|f(x)| \leqslant 2 \cdot 1 \cdot (x \cdot 1 + 1)/x^3 \leqslant 4/x^2$ for $x \geqslant 1$. It follows that B can be made small by choosing N large. Once N is fixed, A can be then made small by picking $|k|$ small.

REFERENCES

1. J. M. Ash and G. V. Welland, Convergence, summability, and uniqueness of multiple trigonometric series, *Bull. Amer. Math. Soc.*, 77 (1971) 123–127. MR 43#812.
2. ______, Convergence, uniqueness, and summability of multiple trigonometric series, *Trans. Amer. Math. Soc.*, 163 (1972) 401–436. MR 45#9057.
3. N. K. Bary, A Treatise on Trigonometric Series, Vol. 2, MacMillan, New York, 1964. MR 30#1347.
4. G. Cantor, Über einen die trigonometrischen Reihen betreffenden Lehrsatz, *Crelles J. für Math.*, 72 (1870) 130–138; also in Gesammelte Abhandlungen, Georg Olms, Hildesheim, 1962, pp. 71–79.
5. ______, Beweis, das eine für jeden reellen Wert von x durch eine trigonometrische Reihe gegebene Funktion $f(x)$ sich nur auf eine einzige Weise in dieser Form darstellen lässt, *Crelles J. für Math.*, 72 (1870) 139–142; also in Gesammelte Abhandlungen, Georg Olms, Hildesheim, 1962, pp. 80–83.
6. B. Connes, Sur les coefficients des séries trigonométriques convergentes sphériquement, *C. R. Acad. Sci. Paris*, Ser A, 283 (1976) 159–161. MR 54#10975.

7. R. Cooke, A Cantor-Lebesgue theorem in two dimensions, *Proc. Amer. Math. Soc.*, 30 (1971), 547–550. MR 43#7847.
8. ______, The Cantor-Lebesgue theorem, this MONTHLY, 86 (1979) 558–565. MR 81b:42019.
9. J. W. Dauben, Georg Cantor: His Mathematics and Philosophy of the Infinite, Harvard Univ. Press, Cambridge, Mass., 1979. MR 80g:01021.
10. D. Men'shov, Sur la convergence uniforme des séries de Fourier (in Russian), *Mat. Sbornik*, 11 (1942) 67–96. MR 3-106.
11. Y. Meyer, Algebraic Numbers and Harmonic Analysis, North-Holland, Amsterdam, and American Elsevier, New York, 1972. MR 58#5579.
12. T. Radó, Subharmonic Functions, Chelsea, New York, 1949.
13. B. Riemann, Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe, Gesammelte Abhandlungen, 2. Aufl., Georg Olms, Hildesheim, 1962, pp. 227–271; also Dover, New York, 1953.
14. K. A. Ross, Elementary Analysis: The Theory of Calculus, Springer-Verlag, New York, 1980. MR 81a:26001.
15. R. Salem, Algebraic Numbers and Fourier Analysis, D. C. Heath, Boston, 1963. MR 28#1169.
16. R. Salem and A. Zygmund, Sur un théorème de Piatetski-Shapiro, *C. R. Acad. Paris*, 240 (1955) 2040–2042. MR 17-150.
17. ______, Sur les ensembles parfaits dissymétriques à rapport constant, ibid *C. R. Acad. Paris*, 240 (1955) 2281–2283. MR 17-150.
18. V. L. Shapiro, Fourier series in several variables, *Bull. Amer. Math. Soc.*, 70 (1964) 48–93. MR 28#1448.
19. ______, Uniqueness of multiple trigonometric series, *Ann. of Math.*, (2) 66 (1957) 467–480. MR 19-854, 1432.
20. E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces, Princeton Univ. Press, Princeton, 1971. MR 46#4102.
21. Ch. J. de la Vallée-Poussin, Sur l'unicité du développement trigonométrique, *Bull. Acad. Roy. de Belg.* (1912) 702–718.
22. S. Verblunsky, On the theory of trigonometric series, I, *Proc. Lon. Math. Soc.*, 34 (1932) 441–456.
23. ______, On the theory of trigonometric series, II, *Proc. Lon. Math. Soc.*, 34 (1932) 457–491.
24. W. H. Young, A note on trigonometrical series, *Mess. for Maths.*, 38 (1909) 44–48.
25. A. Zygmund, Trigonometric series. Vols. 1, 2, 2nd rev. ed., Cambridge Univ. Press, New York, 1959. MR 21#6498.
26. ______, A Cantor-Lebesgue theorem for double trigonometric series, *Studia Math.*, 43 (1972) 171–175. MR 47#711.
27. ______, Notes on the history of Fourier series, Studies in Harmonic Analysis, Studies in Mathematics, Vol. 13, J. M. Ash, ed., Math. Assoc. of America, 1976, pp. 1–19. MR 56#12740.